Advanced Topics in Superstring Theory

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Lecture 1

Particles

- The action for the relativistic massive particle is given by

$$S = -M \int d\tau \sqrt{-\dot{X}^m \dot{X}_m} \tag{1}$$

- The momentum associated to X^m is easily computed to be

$$P_m = \frac{\partial L}{\partial \dot{X}^m} = M \frac{\dot{X}_m}{\sqrt{-\dot{X}^m \dot{X}_m}} \tag{2}$$

- It satisfies the constraint

$$P^2 + M^2 = 0 \tag{3}$$

- Therefore one can write down (1) in its Hamiltonian form as

$$S = \int d\tau \left[P_m \dot{X}^m - \frac{e}{2}(P^2 + M^2) \right] \tag{4}$$

- In this manner, a massless particle is described by

$$S = \int d\tau \left[P_m \dot{X}^m - \frac{e}{2} P^2 \right] \quad (5)$$

- To study the physical spectrum described by (5), one needs to fix the reparametrization symmetry. After gauge-fixing $e = 1$, one finds
- The gauge-fixed action:

$$S = \int d\tau \left[P_m \dot{X}^m - \frac{1}{2} P^2 + b\dot{c} \right] \quad (6)$$

- The BRST operator:

$$Q = cP^2 \quad (7)$$

- As usual, physical states are defined as elements of the BRST-cohomology:

$$\mathcal{H}_{Hilbert} = \frac{Ker(Q)}{Im(Q)} \tag{8}$$

- A general wavefunction in a coordinate representation can be written as

$$\Phi(x,c) = \phi(x) + c\tilde{\phi}(x) \tag{9}$$

- The condition $Q\Phi = 0$ implies that

$$\Box\phi = 0 \tag{10}$$

- And the condition $\delta\Phi = Q\Omega$ implies that

$$\delta\tilde{\phi} = \Box w \tag{11}$$

where $\Omega(x,c)$ is given by

$$\Omega(x,c) = w(x) + c\tilde{w}(x) \tag{12}$$

with w, $\tilde{w}$ being arbitrary gauge parameters.

- Using w one can remove the piece of $\tilde{\phi}(x)$ that does not satisfy $\Box\tilde{\phi} = 0$. The Hilbert space is then described by ϕ, $\tilde{\phi}$ satisfying

$$\Box\phi = 0 \quad , \quad \Box\tilde{\phi} = 0 \tag{13}$$

  where ϕ and $\tilde{\phi}$ are ghost number one and zero fields, respectively.
- We call the solution $\phi(x)$ a physical field solution, and $\tilde{\phi}(x)$ will be called the antifield solution.

Spinning Particles

- The spinning particle action is defined by

$$S = \int d\tau \Big[P_m \dot{X}^m + \frac{1}{2}\psi_m \dot{\psi}^m - \frac{e}{2}P^2 + \chi P \cdot \psi \Big] \tag{14}$$

- Since we are interested in studying superstrings, we will assume $m = 0, 1, \ldots, 9$. In this manner, ψ^m is an $SO(1,9)$ fermionic vector.
- After gauge-fixing $e = 1$, $\chi = 0$, one is left with

$$S = \int d\tau \Big[P_m \dot{X}^m + \frac{1}{2}\psi_m \dot{\psi}^m - \frac{1}{2}P^2 + b\dot{c} + \beta\dot{\gamma} \Big] \tag{15}$$

and the BRST operator

$$Q = cP^2 + \gamma P \cdot \psi - \frac{1}{2}b\gamma^2 \tag{16}$$

- Since $\{\psi^m, \psi^n\} = \eta^{mn}$, one needs to find a representation of the Clifford algebra to define the ground state.
- This can be done in two ways: Breaking $SO(10)$ down to $SU(5)$ and defining a scalar ground state, or preserving 10D Lorentz covariance and using a non-scalar ground state.
- We choose to preserve Lorentz covariance, and so our ground state is described by a 32-component spinor Σ^A which behaves under the action of ψ^m in the usual way

$$\psi^m \Sigma^A = \frac{1}{\sqrt{2}} (\Gamma^m)^A{}_B \Sigma^B \tag{17}$$

  where $(\Gamma^m)^A{}_B$ is a 10D gamma matrix satisfying the standard Clifford algebra.
- A general wavefunction in a coordinate representation is then written down as

$$\Phi^A(x, c, \gamma) = \phi^A_{0,0}(x) + c\phi^A_{1,0}(x) + \gamma\phi^A_{0,1}(x) + c\gamma\phi^A_{1,1}(x) + \dots \tag{18}$$

- Analogously, an arbitrary gauge parameter can be expanded as

$$\Omega^A(x, c, \gamma) \;=\; w^A_{0,0}(x) + c w^A_{1,0}(x) + \gamma w^A_{0,1}(x) + c\gamma w^A_{1,1}(x) + \ldots \tag{19}$$

- Physical states conditions then imply

$$\begin{aligned}
c: & \qquad \Box \phi^A_{0,0} = 0 \\
\gamma: & \qquad \not{k}^A{}_B \phi^B_{0,0} = 0 \\
c\gamma: & \qquad -\frac{1}{\sqrt{2}} \not{k}^A{}_B \phi^B_{1,0} + \Box \phi^A_{0,1} = 0 \\
\gamma^2: & \qquad \frac{1}{\sqrt{2}} \not{k}^A{}_B \phi^B_{0,1} - \frac{1}{2} \phi^A_{1,0} = 0 \\
\vdots &
\end{aligned} \tag{20}$$

- And the gauge transformations read

$$
\begin{aligned}
c : \quad & \delta\phi^A_{1,0} = \Box w^A_{0,0} \\
\gamma : \quad & \delta\phi^A_{0,1} = \frac{1}{\sqrt{2}} \not{k}^A{}_B w^B_{0,0} \\
c\gamma : \quad & \delta\phi^A_{1,1} = \Box w^A_{0,1} - \frac{1}{\sqrt{2}} \not{k}^A{}_B w^B_{1,0} \\
\gamma^2 : \quad & \delta\phi^A_{0,2} = -\frac{1}{2} w^A_{1,0} + \frac{1}{\sqrt{2}} (\not{w}_{0,1})^A \\
\vdots \quad &
\end{aligned}
\tag{21}
$$

- One can then show that the gauge transformations (21) imply that $P^2\phi^A = 0$, $\not{k}^A{}_B \phi^B \neq 0$ are the only requirements describing a non-trivial cohomology. Hence, the Hilbert space is described by an infinite set of spin-$\frac{1}{2}$ fields: $\phi^A_{0,0}$, $\phi^A_{1,0}$, $\ldots$.

BV Description of Super-Yang-Mills

- The BV or antifield formalism allows us to quantize systems constrained with reducible or irreducible symmetries.
- The basic idea is to add a ghost variable (c^A) for each symmetry present in the theory. The whole set of matter and ghost variables will be denoted by $\Phi^I = (\phi^i, c^A)$.
- Next, one adds an antifield for each field in Φ^I. The whole set of antifields will be denoted by $\Phi^*_I = (\phi^*_i, c^*_A)$. Their ghost number charges are related to each other by: $\mathrm{gh}(\Phi^I) + \mathrm{gh}(\Phi^*_I) = -1$.
- These ingredients allow us introduce the so-called antibracket:

$$(A, B) \;=\; \frac{\delta_R A}{\delta \Phi^I}\frac{\delta_L B}{\delta \Phi^*_I} - \frac{\delta_R A}{\delta \Phi^*_I}\frac{\delta_L B}{\delta \Phi^I} \tag{22}$$

- The master equation is then defined as

$$(\mathcal{S}, \mathcal{S}) \;=\; 0 \tag{23}$$

- A solution to the master equation is called a master action.
- Since the master action contains the original action, it is bosonic and has ghost number zero. Therefore the master action satisfies

$$\frac{\delta_L \mathcal{S}}{\delta \Phi^I} \frac{\delta_L \mathcal{S}}{\delta \Phi^*_I} = 0 \tag{24}$$

- To solve (24), one can expand S into a sum of terms with different antifield number. Explicitly,

$$\mathcal{S} = S_0 + S_1 + S_2 + \dots \tag{25}$$

where the subscript stands for the number of antifields present in S_i.
- Therefore, one has

$$\frac{\delta_L S_0}{\delta \Phi^I} \frac{\delta_L S_1}{\delta \Phi^*_I} = 0 \tag{26}$$

$$\frac{\delta_L S_0}{\delta \Phi^I} \frac{\delta_L S_2}{\delta \Phi^*_I} + \frac{\delta_L S_1}{\delta \Phi^I} \frac{\delta_L S_1}{\delta \Phi^*_I} = 0 \tag{27}$$

$$\vdots$$

- For closed gauge algebras ($[\delta_{\alpha_1}, \delta_{\alpha_2}] = \delta_{\alpha_3}$), the only non-vanishing terms are given by S_0, S_1. S_0 is the original action, and S_1 takes the form

$$S_1 \quad = \quad \phi_i^* R^i{}_A[\phi] c^A + c_A^* f_{BC}{}^A c^B c^C + b_A^* h^A \tag{28}$$

where $f_{BC}{}^A$ are the structure constants of the gauge group, and $R^i{}_A[\phi]$ is the gauge transformations of the matter fields:

$$\delta\phi^i \quad = \quad R^i{}_A[\phi]\epsilon^A \tag{29}$$

and ϵ^A is an infinitesimal gauge parameter.

- We are now ready to apply this for our super-Yang-Mills action which reads

$$S_0 \quad = \quad \int d^{10}x Tr\Big[-\frac{1}{4}F^{mn}F_{mn} + \frac{i}{2}(\chi\gamma^m\nabla_m\chi)\Big] \tag{30}$$

- Using the standard gauge transformations for the gluon and gluino fields one can compute ${R^i}_A[\phi]$. Therefore,

$$\begin{aligned} R^{ac}_m(y,x) &= [-\delta^{ac}\partial_m + f^{abc}A^b_m(y)]\delta(y-x) && (31)\\ R^{\alpha\, ac}(y,x) &= f^{abc}\chi^{\alpha\, b}(y)\delta(y-x) && (32)\end{aligned}$$

where $a, b, \ldots$ are Lie algebra indices, and $[T^a, T^b] = f^{abc}T^c$.

- Hence, the master action reads

$$\begin{aligned}\mathcal{S} &= \int d^{10}x\, Tr\Big[-\frac{1}{4}F^{mn}F_{mn} + \frac{i}{2}(\chi\gamma^m\nabla_m\chi) + ia^*_m\nabla^m c \\ &\quad -i\chi^*_\alpha\{c,\chi^\alpha\} - ic^*cc\Big] && (33)\end{aligned}$$

- The equation motion following from this action for the abelian case read

$$\partial^m F_{mn} = 0 \ , \ (\gamma^m)_{\alpha\beta}\partial_m \chi^\beta = 0 \ , \ \partial^m a^*_m = 0 \ , \ \partial_m c = 0 \qquad (34)$$

- The gauge transformations associated to the field and antifield immediately follow from the master equation. Indeed, one can show that

$$\begin{aligned} \delta_\sigma \Phi^I &= \sigma^K \frac{\delta^2 \mathcal{S}}{\delta\Phi^K \delta\Phi^*_I} \\ \delta_\sigma \Phi^*_I &= \sigma^K \frac{\delta^2 \mathcal{S}}{\delta\Phi^K \delta\Phi^I} \end{aligned} \qquad (35)$$

leave invariant the master action.

- For instance, by choosing the direction of σ^K along c^a, one finds (for the abelian case)

$$\delta A^{m\,a} = \partial_m \sigma^a \qquad (36)$$

- Similarly, one gets for the other fields

$$\delta_\lambda a^*_m = 0 \ , \ \delta_\rho \chi^*_\alpha = (\gamma^m)_{\alpha\beta}\partial_m \rho^\beta \ , \ \delta_\epsilon c* = \partial^m \epsilon_m \qquad (37)$$

Strings

- The string action is defined by

$$S = -\int d\tau d\sigma \left[\sqrt{(\dot{X}\cdot X')^2 - (\dot{X}^2)(X')^2} \right] \tag{38}$$

where we conveniently fix $T_s = 1$, and $X'^m = \frac{\partial}{\partial\sigma} X^m$.

- The momentum associated to X^m is then found to be

$$P_m = \frac{\partial L}{\partial \dot{X}^m} = \frac{(X')^2 \dot{X}_m - (\dot{X}\cdot X')X'_m}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2 (X')^2}} \tag{39}$$

- Therefore, the system is constrained by the relations

$$P\cdot X' = 0 \quad , \quad P^2 + (X')^2 = 0 \tag{40}$$

- They can be equivalently rewritten as

$$(P^m \pm X'^m)^2 = 0 \tag{41}$$

- Thus, the first-order form of the Nambu-Goto action is given by

$$
\begin{aligned}
S &= \int d\tau d\sigma \Big[2P_m \dot{X}^m + \frac{e}{2}(P_m + X'_m)(P^m + X'^m) \\
&\quad + \frac{\bar{e}}{2}(P_m - X'_m)(P^m - X'^m) \Big]
\end{aligned} \tag{42}
$$

where e, $\bar{e}$ are Lagrange multipliers enforcing the Virasoro constraints.

- The gauge transformations generated by the Virasoro constraints read

$$
\begin{aligned}
\delta_{\epsilon} X^m(\sigma) &= [\int d\sigma' \epsilon \frac{1}{2}(P + X')^2(\sigma'), X^m(\sigma)] = -\epsilon(P + X')^m(\sigma) \\
\delta_{\bar{\epsilon}} X^m(\sigma) &= [\int d\sigma' \bar{\epsilon} \frac{1}{2}(P - X')^2(\sigma'), X^m(\sigma)] = -\bar{\epsilon}(P - X')^m(\sigma) \\
\delta_{\epsilon} P_m(\sigma) &= [\int d\sigma' \epsilon \frac{1}{2}(P + X')^2(\sigma'), P_m(\sigma)] = -\partial_{\sigma}(\epsilon(P + X')^m)(\sigma) \\
\delta_{\bar{\epsilon}} P_m(\sigma) &= [\int d\sigma' \bar{\epsilon} \frac{1}{2}(P - X')^2(\sigma'), P_m(\sigma)] = \partial_{\sigma}(\bar{\epsilon}(P - X')^m)(\sigma)
\end{aligned} \tag{43}
$$

- One can also show that the Virasoro algebra is non-abelian since

$$[\frac{1}{2}(P \pm X')^2(\sigma), \frac{1}{2}(P \pm X')^2(\sigma')] = \mp 2\partial_{\sigma'}(\delta(\sigma - \sigma'))(P \pm X')^m(\sigma) (P \pm X')_m(\sigma')$$

- It is also not hard to show that the Virasoro constraints induce the following transformations for the Lagrange multipliers

$$\delta_\epsilon e = \dot{\epsilon} + e\partial_\sigma \epsilon - 2\epsilon\partial_\sigma e \quad , \quad \delta_{\bar{\epsilon}}\bar{e} = \dot{\bar{\epsilon}} - \bar{e}\partial_\sigma\bar{\epsilon} + 2\bar{\epsilon}\partial_\sigma\bar{e} \tag{44}$$

- To quantize this model in a Lorentz covariant manner we choose $e = \bar{e} = -1$, then

$$S = \int d\tau \Big[2P_m\dot{X}^m - X'^2 - P^2 + b(\dot{c} - c') + \bar{b}(\dot{\bar{c}} + \bar{c}')\Big] \tag{45}$$

- In this manner, the BRST operator takes the form

$$Q = \int d\sigma \Big[c\frac{(P+X')^2}{2} + \bar{c}\frac{(P-X')^2}{2} + 2bcc' - 2\bar{b}\bar{c}\bar{c}'\Big] \tag{46}$$

- Using the e.o.m for P^m and introducing the conventional notation

$$\partial_z = \partial_\tau + \partial_\sigma \quad , \quad \bar{\partial}_{\bar{z}} = \partial_\tau - \partial_\sigma \tag{47}$$

one is left with the usual bosonic closed string model

$$\begin{aligned} S &= \int dz d\bar{z} \Big[\partial X^m \bar{\partial} X_m + b\bar{\partial} c + \bar{b} \partial \bar{c} \Big] && (48) \\ Q &= \int dz (c \frac{\partial X^m \partial X_m}{2} + bc\partial c) + \int d\bar{z} (\bar{c} \frac{\bar{\partial} X^m \bar{\partial} X_m}{2} + \bar{b}\bar{c}\bar{\partial}\bar{c}) && (49) \end{aligned}$$

- For simplicity, we will focus on the open case which is described by

$$\begin{aligned} S &= \int dz d\bar{z} \Big[\partial X^m \bar{\partial} X_m + b\bar{\partial} c \Big] && (50) \\ Q &= \int dz (c\partial X^m \partial X_m + bc\partial c) && (51) \end{aligned}$$

- As is well-known the target space dimension must be 26 in order for the BRST operator to be nilpotent at quantum level.

- The physical spectrum is then found by using standard OPE techniques or oscillator mode expansions. For simplicity we will use the latter.
- To this end, we should remember that after mapping the cylinder into the complex plane through $z = e^{-iw}$ with $w = \sigma + i\tau$, the worldsheet fields can be written as

$$X^m(z,\bar{z}) = x_0^m + p^m log|z|^2 + \sum_{k\neq 0} \frac{a_k^m}{k}(z^{-k} + \bar{z}^{-k}) \tag{52}$$

$$b(z) = \sum_k \frac{b_k}{z^{k+2}} \tag{53}$$

$$c(z) = \sum_k \frac{c_k}{z^{k-1}} \tag{54}$$

- As usual, the ground state is defined to be annihilated by all the modes a_k^m, c_i, b_j with $k > 0$, $i > 1$, $j > -2$, respectively.

- The most general wavefunction then reads

$$\Phi(x_0, a^m_{-1}, a^m_{-2} \ldots, b_{-2}, b_{-3}, \ldots, c_1, c_0, \ldots) = \prod_{k,i,j} a^{N_a}_{-k} b^{N_b}_{-i} c^{N_c}_j e^{ik.x_0}|0> \quad (55)$$

- The zero mode of the Virasoro constraint (L_0) in the BRST-charge then implies the familiar mass formula

$$k^2 + \sum_{n=1}^{\infty} n(N_{bn} + N_{cn} + \sum_{m=0}^{25} N^a_{m,n}) = 0 \quad (56)$$

- After splitting the spectrum into diferent mass levels, one finds

$$k^2 = -1 \quad : \quad c_1 e^{ik.x_0}|0> \quad (57)$$

$$k^2 = 0 \quad : \quad e^{ik.x_0}|0>, \; c_1 a^m_{-1} e^{ik.x_0}|0>, \; c_0 e^{ik.x_0}|0>, \; c_{-1} c_1 e^{ik.x_0}|0>, \; c_0 c_1 a^m_{-1} e^{ik.x_0}|0>, \; c_0 c_{-1} c_1 e^{ik.x_0}|0> \quad (58)$$

$$k^2 = 1 \quad : \quad b_{-2} c_1 e^{ik.x_0}|0> + \ldots \quad (59)$$

$$k^2 = 2 \quad : \quad b_{-2} e^{ik.x_0}|0> + \ldots \quad (60)$$

- Since we will be interested in studying the scattering of masless particles, let us restrict our study to the massless level of the string spectrum.
- The state-operator correspondence then dictates

$$\begin{aligned} \Phi^{k^2=0}(x,c) &= C + c\partial X^m A_m + \partial c B + c\partial^2 c D \\ &+ c\partial c\partial X^m \tilde{A}_m + c\partial c\partial^2 c\tilde{C} \end{aligned} \quad (61)$$

- The full BRST-cohomology condition requires that $\partial_m C = 0$, and

$$\begin{array}{llll} \partial^m A_m = B & \delta A_m = \partial_m \Lambda & \Box A_m = \partial_m B & \delta B = \Box\Lambda \\ \partial^m \tilde{A}_m = \Box D & \delta\tilde{A}_m = \Box S_m - \partial_m\phi & \delta D = -\phi + \partial^m S_m & \delta\tilde{C} = \partial^m\Omega_m \end{array} \quad (62)$$

- These are exactly the fields, ghosts and their respective antifields of the Batalin-Vilkovisky description of Yang-Mills (in 26D).

Spinning Strings

- The spinning string action is defined by

$$
\begin{aligned}
S \;=\; & \int d\tau d\sigma \Big[2P_m \dot{X}^m + \psi^m \dot{\psi}_m + \bar{\psi}^m \dot{\bar{\psi}}_m + \frac{e}{2}\Big((P+X')^2 \\
& + 2\psi^m \psi'_m \Big) + \frac{\bar{e}}{2}\Big((P-X')^2 - 2\bar{\psi}^m \bar{\psi}'_m \Big) + \chi[\psi \cdot (P+X')] \\
& + \bar{\chi}[\bar{\psi} \cdot (P-X')] \Big] \qquad (63)
\end{aligned}
$$

- The periodicity properties of the matter fermionic variables give rise to the so-called (R) Ramond and (NS) Neveu-Schwarz sectors, defined as

$$
\begin{array}{llll}
\text{C.S}: & \psi^m(\sigma+2\pi) = \psi^m(\sigma) \quad R, & \psi^m(\sigma+2\pi) = -\psi^m(\sigma) & NS \\
 & \bar{\psi}^m(\sigma+2\pi) = \bar{\psi}^m(\sigma) \quad R, & \bar{\psi}^m(\sigma+2\pi) = -\bar{\psi}^m(\sigma) & NS \\
\text{O.S}: & \psi^m(0) = \bar{\psi}^m(0) \quad , & \psi^m(\pi) = \bar{\psi}^m(\pi) & R \\
 & \psi^m(0) = \bar{\psi}^m(0) \quad , & \psi^m(\pi) = -\bar{\psi}^m(\pi) & NS
\end{array}
$$

- As done before, we fix $e = \bar{e} = -1$, $\chi = \bar{\chi} = 0$. After solving the e.o.m for P^m and introduce complex derivatives, one arrives at

$$
\begin{aligned}
S &= \int d^2z \Big[\partial X^m \bar{\partial} X_m + \psi^m \bar{\partial}\psi_m + \bar{\psi}^m \partial \bar{\psi}_m \\
&\quad + b\bar{\partial}c + \bar{b}\partial\bar{c} + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma}\Big] \qquad (64)
\end{aligned}
$$

- Using a similar procedure as the one discussed in the bosonic case, the BRST operator for the spinning string is found to be

$$
\begin{aligned}
Q &= \int dz \Big[c\Big(\frac{\partial X \cdot \partial X + \psi \cdot \partial\psi}{2}\Big) + \gamma(\partial X \cdot \psi) + bc\partial c \\
&\quad + c\Big(\partial\beta\gamma - \frac{3}{2}\partial(\beta\gamma)\Big) + b\gamma^2\Big] + c.c \qquad (65)
\end{aligned}
$$

- Henceforth we will concentrate on the open string model.
- As before, we will make a mode expansion to analyze the physical content of the theory.
- The modes for X, b, c were already studied. Therefore, the new variables possess the following mode expansions:

$$\psi^m(z) = \sum_{\mathbb{Z}+\nu} \frac{\psi_k^m}{z^{k+\frac{1}{2}}} \tag{66}$$

$$\beta(z) = \sum_{\mathbb{Z}+\nu} \frac{\beta_k}{z^{k+\frac{3}{2}}} \tag{67}$$

$$\gamma(z) = \sum_{\mathbb{Z}+\nu} \frac{\gamma_k}{z^{k-\frac{1}{2}}} \tag{68}$$

  where $\nu = 0, \frac{1}{2}$ if the boundary condition is R, NS respectively.
- Then, the ground state will be different for each sector. For instance, the NS ground state is required to be annihilated by ψ_k^m, β_i, γ_j with $k > 0$, $i > -\frac{3}{2}$,$j > \frac{1}{2}$. On the other hand, the R ground state is annihilated by ψ_k^m, β_i, γ_j with $k > 0$, $i > -1$,$j > 1$.

- To explicitly construct the ground states in both sectors one needs to use bosonization techniques. For reasons of time, we will omit this construction and just focus on the physical spectrum.
- The L_0 mode in the BRST-charge imposes again the mass formula for each state in both sectors. For simplicity, let us just analyze the NS sector.
- The spectrum then reads

$$
\begin{aligned}
k^2 = -\frac{1}{2} \quad &: \quad \gamma_{\frac{1}{2}}|0>\,,\ c_1\psi^m_{-\frac{1}{2}}|0> && (69)\\
k^2 = 0 \quad &: \quad |0>\,,\ c_1 a^m_{-1}|0>\,,\ \gamma_{\frac{1}{2}}\psi^m_{-\frac{1}{2}}|0>\,,\ c_0|0>\,,\\
&\quad c_0 c_1 a^m_{-1}|0>\,,\ c_0 c_{-1} c_1|0>\,,\ c_0\gamma_{\frac{1}{2}}\gamma_{-\frac{1}{2}}|0>\,,\ \ldots && (70)\\
&\vdots
\end{aligned}
$$

- As before, one can write down the vertex operators for these states as

$$\Phi^{k^2=-\frac{1}{2}}(x,\psi,c,\gamma) = \gamma\phi + c(\psi\cdot k)\phi \tag{71}$$

$$\Phi^{k^2=0}(x,\psi,c,\gamma) = \gamma\psi^m A_m(x) + c\partial X^m A_m + c\psi^m\psi^n F_{mn} + \text{ghosts} + \text{antifields} \tag{72}$$

- With a little more effort the R sector can also be found.
- In order to realize spacetime supersymmetry one needs to impose by hand the so-called GSO projection. This operation removes the tachyon and gives us a well-defined quantum mechanically theory. However it makes life complicated when computing loop amplitudes (sums over spin structures).
- The resulting massless spectrum describes the Batalin-Vilkovisky description of super-Yang-Mills (in 10D).

Summary

- The quantization of the particle model is described by two scalars of opposite statistics $\phi(x)$, $\tilde{\phi}(x)$ satisfying the massless KG e.o.m.
- The quantization of the spinning particle is described by an infinite set of spin-$\frac{1}{2}$ fields satisfying the Weyl e.o.m.
- The BV description of super-Yang-Mills is simple because of the closure of the gauge algebra and it incorporates all the symmetries of the theory through the introduction of ghosts and antifields to the original model.
- The bosonic string spectrum possesses a tachyon and it describes the BV formulation of Yang-Mills in its massless level.
- The superstring, i.e the spinning string after GSO projection, does not have tachyons in its spectrum and it describes the BV formulation of super-Yang-Mills in its massless level.

Lecture 2

10D Brink-Schwarz superparticle

- The 10D Brink-Schwarz superparticle action is defined by

$$S = \int d\tau \left[P_m \Pi^m - \frac{e}{2} P^2 \right] \tag{73}$$

where $\Pi^m = \dot{X}^m + i(\theta\gamma^m\dot{\theta})$.

- In 10D, the spinor representation is 32-dimensional and reducible. The two irrep. are 16-dimensional and they are called MW rep. Depending on its eigenvalue under Γ^{11}, spinors are classified as being chiral (χ^α) or antichiral (χ_α).
- The action (73) is invariant under worldline reparametrizations, (global) SUSY transformations

$$\delta\theta^\alpha = \epsilon^\alpha \ , \ \ \delta X^m = -i(\epsilon\gamma^m\theta) \ , \ \ \delta P^m = 0 \ , \ \ \delta e = 0 \tag{74}$$

and the so-called (local) kappa symmetry

$$\delta\theta^\alpha = (\gamma^m\kappa)^\alpha P_m \ , \ \delta X^m = i(\delta\theta\gamma^m\theta) \ , \ \delta P^m = 0 \ , \ \delta e = -4i\dot{\theta}^\alpha\kappa_\alpha \tag{75}$$

- The momentum associated to the coordinate θ^α is found to be

$$p_\alpha = \frac{\partial L}{\partial \dot{\theta}^\alpha} = -i(\gamma^m \theta)_\alpha P_m \tag{76}$$

- Therefore we have a constrained system. The constraints read $d_\alpha = 0$ with

$$d_\alpha = p_\alpha + i(\gamma^m \theta)_\alpha P_m \tag{77}$$

- The algebra satisfied by these constraints takes the form

$$\{d_\alpha, d_\beta\} = -2(\gamma^m)_{\alpha\beta} P_m \tag{78}$$

- One can show that the kappa symmetry transformations are generated by $K^\alpha = -i(\gamma^m d)^\alpha P_m$, which satisfies the algebra

$$\{K^\alpha, K^\beta\} = 2(\gamma^m)^{\alpha\beta} P_m P^2 \tag{79}$$

- Since $P^2 = 0$, d_α contains 8 first-class and 8 second-class constraints.
- This can easily be seen by chossing a particular reference frame where $P^m = (P^+, 0, \ldots, 0)$. Then, using the 10D gamma matrices

$$(\gamma^+)_{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{2} \end{pmatrix} \ , \ (\gamma^-)_{\alpha\beta} = \begin{pmatrix} -\sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \tag{80}$$

and the splitting of a 10D spinor into its $SO(8)$ components, namely

$$\chi^\alpha = \begin{pmatrix} \chi^a \\ \chi^{\dot{a}} \end{pmatrix} \tag{81}$$

where $a, \dot{a} = 1, \ldots, 8$ are the Weyl and anti-Weyl spinor representations of $SO(8)$, the constraint algebra can be written as

$$\{d_a, d_b\} = -2\sqrt{2}\delta_{ab}P^+ \ , \ \{d_a, d_{\dot{b}}\} = 0 \ , \ \{d_{\dot{a}}, d_{\dot{b}}\} = 0 \tag{82}$$

- Then $d_{\dot{a}}$ are first-class constraints and d_a are second-class constraints.
- It turns out that there is no simple way to separate these constraints out in a Lorentz covariant manner.

- However, we can compute the physical spectrum in a simple way by gauge-fixing the kappa symmetry.
- In this manner, the semi-light cone gauge is defined by $(\gamma^+\theta)_\alpha = 0$. A simple way of seeing how this gauge choice can always be implemented is by choosing the reference frame $P^m = (P^+, 0, \ldots, 0)$ and performing a kappa transformation on θ^α:

$$\theta'^\alpha = \theta^\alpha + \delta_\kappa \theta^\alpha = \theta^\alpha - (\gamma^-\kappa)^\alpha P^+ = \theta^\alpha + \frac{1}{2}(\gamma^-\gamma^+\theta) \tag{83}$$

where $\kappa = -\frac{1}{2P^+}(\gamma^+\theta)^\alpha$. Then, one writes down

$$\theta^\alpha = -\frac{1}{2}(\gamma^+\gamma^-\theta)^\alpha - \frac{1}{2}(\gamma^-\gamma^+\theta)^\alpha \tag{84}$$

Putting all together we see that $\theta'^\alpha = -\frac{1}{2}(\gamma^+\gamma^-\theta)$ which satisfies $(\gamma^+\theta)_\alpha = 0$ since $(\gamma^+)^2 = 0$.

- Then, the gauge-fixed BS action takes the form

$$S = \int d\tau \left[P_m \dot{X}^m - i(\theta\gamma^- \dot{\theta})P^+ - \frac{e}{2}P^2 \right] \tag{85}$$

- One then defines the variables S^a: as $S^a = 2^{\frac{1}{4}}(P^+)^{\frac{1}{2}}(\gamma^-\theta)^a$, and rewrites the action in the form

$$S = \int d\tau \left[P_m \dot{X}^m + \frac{i}{2} S^a \dot{S}^a - \frac{e}{2}P^2 \right] \tag{86}$$

- The momentum associated to S^a is then given by

$$p_a = \frac{\partial L}{\partial \dot{S}^a} = -\frac{i}{2} S_a \tag{87}$$

which defines the constraint

$$\tilde{d}_a = p_a + \frac{i}{2} S_a \tag{88}$$

- Using the standard Poisson brackets for p_a and S^a one finds

$$\{\tilde{d}_a, \tilde{d}_b\} = \delta_{ab} \tag{89}$$

- Using the usual Dirac procedure to quantize systems with second class constraints, namely

$$\{A, B\}_D = \{A, B\}_P - \sum_{e,f}\{A, \phi^e\}_P C^{-1}_{ef}\{\phi^f, B\}_P \qquad (90)$$

where $\{\phi^e, \phi^f\}_P = C^{ef}$, we get in our case

$$\{S^a, S^b\}_D = \delta^{ab} \qquad (91)$$

- Using the triality property of $SO(8)$, that is

$$\begin{aligned}
(\sigma^i)_{a\dot{a}}(\sigma^j)_{b\dot{a}} + (\sigma^j)_{a\dot{a}}(\sigma^i)_{b\dot{a}} &= 2\delta^{ij}\delta_{ab} \\
(\sigma^i)_{a\dot{a}}(\sigma^j)_{a\dot{b}} + (\sigma^j)_{a\dot{a}}(\sigma^i)_{a\dot{b}} &= 2\delta^{ij}\delta_{\dot{a}\dot{b}} \\
(\sigma^i)_{a\dot{a}}(\sigma^i)_{b\dot{b}} + (\sigma^i)_{a\dot{a}}(\sigma^i)_{b\dot{b}} &= 2\delta_{ab}\delta_{\dot{a}\dot{b}}
\end{aligned} \qquad (92)$$

one can show that the physical states realizing (91) are given by

$$S_a|\dot{a}>= \frac{1}{\sqrt{2}}(\sigma^i)_{a\dot{a}}|i> \quad , \quad S_a|i>= \frac{1}{\sqrt{2}}(\sigma_i)_{a\dot{a}}|\dot{a}> \qquad (93)$$

- These are the light-cone gauge equations of motion of 10D super-Maxwell.

10D Green-Schwarz Superstrings

- The 10D Green-Schwarz superstring action is given by

$$
\begin{aligned}
S &= \int d\tau d\sigma \Big[P_m \Pi_0^m + B^{flat}_{MN} \partial_{[0} Z^M \partial_{1]} Z^N \\
&\quad + e(P^m + \Pi_1^m)(P_m + \Pi_{1\,m}) + \bar{e}(P^m - \Pi_1^m)(P_m - \Pi_{1\,m}) \Big]
\end{aligned} \tag{94}
$$

where $Z^M = (X^m, \theta_L^\alpha, \theta_R^{\hat{\alpha}})$, and

$$
\Pi_\mu = \partial_\mu X^m - \frac{i}{2}(\theta_L \gamma^m \partial_\mu \theta_L) - \frac{i}{2}(\theta_R \gamma^m \partial_\mu \theta_R) \ , \ \ \mu = 1, 2. \tag{95}
$$

- Also, e, $\bar{e}$ are Lagrange multipliers enforcing their respective constraints and B^{flat}_{MN} is the flat value of the type II supergravity 2-form superfield, that is

$$
B_{\alpha\, m} = i(\gamma_m \theta_L)_\alpha \ , \ \ B_{\hat{\alpha}\, m} = -i(\gamma_m \theta_R)_{\hat{\alpha}} \ , \ \ B_{\alpha\hat{\beta}} = (\gamma^m \theta_L)_\alpha (\gamma_m \theta_R)_{\hat{\beta}} \tag{96}
$$

- Explicitly,

$$
\begin{aligned}
S \;=\; \int d\tau d\sigma \Big[& 2P_m \Pi_0^m + i\partial_1 X^m \Big[(\theta_L \gamma_m \partial_0 \theta_L) - (\theta_R \gamma_m \partial_0 \theta_R) \Big] \\
& - i\partial_0 X^m \Big[(\theta_L \gamma_m \partial_1 \theta_L) - (\theta_R \gamma_m \partial_1 \theta_R) \Big] \\
& + (\theta_L \gamma^m \partial_0 \theta_L)(\theta_R \gamma_m \partial_1 \theta_R) - (\theta_L \gamma^m \partial_1 \theta_L)(\theta_R \gamma_m \partial_0 \theta_R) \\
& + \frac{e}{2}(P^m + \Pi_1^m)(P_m + \Pi_{1\,m}) + \frac{\bar{e}}{2}(P^m - \Pi_1^m)(P_m - \Pi_{1\,m}) \Big]
\end{aligned}
\tag{97}
$$

- This action is invariant under the $N = 2$ SUSY transformations

$$
\delta\theta_L^\alpha = \epsilon_L^\alpha \ , \ \ \delta\theta_R^{\hat{\alpha}} = \epsilon_R^{\hat{\alpha}} \ , \ \ \delta X^m = \frac{i}{2}(\epsilon_L \gamma^m \theta_L) + \frac{i}{2}(\epsilon_R \gamma^m \theta_R) \tag{98}
$$

and $\delta P^m = \delta e = 0$.

- To prove this one needs to use the 10D gamma matrix identity $(\gamma^m)_{(\alpha\beta}(\gamma_m)_{\delta)\epsilon} = 0$.

- This very same identity allows us to show that the GS superstring action is invariant under the kappa transformations

$$\begin{aligned}\delta\theta_L^{\alpha} &= (\gamma^m\kappa_L)^{\alpha}(P_m - \Pi_{1\,m}) & \delta\theta_R^{\hat\alpha} &= (\gamma^m\kappa_R)^{\hat\alpha}(P_m + \Pi_{1\,m})\\ \delta X^m &= -\frac{i}{2}(\delta\theta_L\gamma^m\theta_L) - \frac{i}{2}(\delta\theta_R\gamma^m\theta_R) & \delta e &= 4i\kappa_L\partial_R\theta_L\\ \delta P^m &= i(\delta\theta_L\gamma^m\partial_1\theta_L) - i(\delta\theta_R\gamma^m\partial_1\theta_R) & \delta\bar e &= 4i\kappa_R\partial_L\theta_R \end{aligned} \tag{99}$$

  where $\partial_R = \partial_0 - \bar e\partial_1$, $\partial_L = \partial_0 + e\partial_1$.

- Indeed one can show that the action transforms as

$$\begin{aligned}\delta S &= \int d\tau d\sigma\Big[-2i(\delta\theta_L\gamma_m\partial_R\theta_L)(P^m - \Pi_1^m)\\ &\quad -2i(\delta\theta_R\gamma_m\partial_L\theta_R)(P^m + \Pi_1^m) + \frac{\delta e}{2}(P+\Pi_1)^2 + \frac{\delta\bar e}{2}(P-\Pi_1)^2\Big]\end{aligned} \tag{100}$$

  It is not hard to see that the transformations for the Lagrange multipliers given in (99) will exactly cancel the first two terms.

- The momenta associated to the fermionic coordinates then read

$$p_{L\,\alpha} = \frac{\partial L}{\partial \dot{\theta}_L^{\alpha}} = i(\gamma^m \theta_L)_\alpha P_m - B^{flat}_{\alpha\,N} \partial_1 Z^N \tag{101}$$

$$p_{R\,\hat{\alpha}} = \frac{\partial R}{\partial \dot{\theta}_R^{\hat{\alpha}}} = i(\gamma^m \theta_R)_{\hat{\alpha}} P_m - B^{flat}_{\hat{\alpha}\,N} \partial_1 Z^N \tag{102}$$

- Then, the system is constrained by

$$d_{L\,\alpha} = p_{L,\alpha} - i(\gamma^m \theta_L)_\alpha P_m + B^{flat}_{\alpha\,N} \partial_1 Z^N \tag{103}$$

$$d_{R\,\hat{\alpha}} = p_{R,\hat{\alpha}} - i(\gamma^m \theta_R)_{\hat{\alpha}} P_m + B^{flat}_{\hat{\alpha}\,N} \partial_1 Z^N \tag{104}$$

- Explicitly, they read

$$\begin{aligned} d_{L\,\alpha} &= p_{L\,\alpha} - i(\gamma^m\theta_L)_\alpha(P_m - \Pi_{1\,m}) + (\gamma^m\theta_L)_\alpha(\theta_L\gamma_m\partial_1\theta_L) \\ d_{R\,\hat{\alpha}} &= p_{R\,\hat{\alpha}} - i(\gamma^m\theta_R)_{\hat{\alpha}}(P_m + \Pi_{1\,m}) - (\gamma^m\theta_R)_{\hat{\alpha}}(\theta_R\gamma_m\partial_1\theta_R) \end{aligned} \tag{105}$$

- A straightforward computation teaches us that

$$\begin{aligned} \{d_{L\,\alpha}, d_{L\,\beta}\} &= -(P^m - \Pi_1^m)(\gamma_m)_{\alpha\beta} \\ \{d_{R\,\hat{\alpha}}, d_{R\,\hat{\beta}}\} &= -(P^m + \Pi_1^m)(\gamma_m)_{\hat{\alpha}\hat{\beta}} \\ \{d_{L\,\alpha}, d_{R\,\hat{\alpha}}\} &= 0 \end{aligned} \tag{106}$$

- Since $(P \pm \Pi_1)^2 = 0$, we will have 16 first-class and 16 second-class constraints. There is not simple way of separating them out in a Lorentz covariant manner.
- However, we can use the light-cone gauge to study the physical spectrum.

- Let us first write down the action in conformal gauge $e = \bar{e} = -1$:

$$
\begin{aligned}
S \;=\; & \int d^2z \Big[\partial X^m \bar{\partial} X_m - i\partial X^m (\theta_L \gamma_m \bar{\partial}\theta_L) - i\bar{\partial} X^m (\theta_R \gamma_m \partial \theta_R) \\
& -\frac{1}{2}(\theta_L \gamma^m \bar{\partial}\theta_L)\Big[(\theta_L\gamma_m\partial\theta_L) + (\theta_R\gamma_m\partial\theta_R)\Big] \\
& -\frac{1}{2}(\theta_R \gamma^m \partial\theta_R)\Big[(\theta_L\gamma_m\bar{\partial}\theta_L) + (\theta_R\gamma_m\bar{\partial}\theta_R)\Big]\Big] \qquad (107)
\end{aligned}
$$

- Then, we use the residual symmetry + kappa transformations to fix the light-cone gauge:

$$
X^+ = x^+ + p^+\tau \;,\; (\gamma^+\theta_L)_\alpha = 0 \;,\; (\gamma^+\theta_R)_{\hat{\alpha}} = 0 \qquad (108)
$$

- In this gauge, all the quartic terms in θ in (107) vanish and we are left with

$$
S \;=\; \int d^2z \Big[\partial X^i \bar{\partial} X^i + \frac{1}{2} S_L^a \bar{\partial} S_L^a + \frac{1}{2} S_R^{\hat{a}} \partial S_R^{\hat{a}} \Big] \qquad (109)
$$

where $S_L^a = 2^{\frac{3}{4}} i \sqrt{ip^+}\theta_L^a$, $S_R^{\hat{a}} = 2^{\frac{3}{4}} i \sqrt{ip^+}\theta_R^{\hat{a}}$.

- If the chiralities of the $SO(8)$ spinors are the same (opposite), the superstring is Type IIB (Type IIA).
- After performing a Wick rotation and making a proper assignment of conformal weights to the free variables in (109), one can expand in modes as we did when we studied the spinning particle. Therefore,

$$\partial X^i(z) = \sum_n \frac{a_n^i}{z^{n+1}}, \qquad \bar{\partial} X^i(\bar{z}) = \sum_n \frac{\bar{a}_n^i}{\bar{z}^{n+1}}$$
$$S_L^a(z) = \sum_n \frac{S_n^a}{z^{n+\frac{1}{2}}}, \qquad S_R^{\hat{a}}(\bar{z}) = \sum_n \frac{\bar{S}_n^{\hat{a}}}{\bar{z}^{n+\frac{1}{2}}} \tag{110}$$

- It is not hard to see the modes will satisfy the following algebras:

$$[x^i, p^j] = \delta^{ij} \qquad [a_m^i, a_n^j] = m\delta^{ij}\delta_{m,-n}$$
$$\{S_m^a, S_n^b\} = \delta^{ab}\delta_{m,-n} \qquad \{\bar{S}_m^{\hat{a}}, \bar{S}_n^{\hat{b}}\} = \delta^{\hat{a}\hat{b}}\delta_{m,-n} \tag{111}$$

- To compute the mass spectrum, one should first compute the momentum density. Using the standard Noether procedure, one finds

$$J^m = i\partial X^m + (\theta_R \gamma^m \partial \theta_R) \quad , \quad \bar{J}^m = i\bar{\partial} X^m + (\theta_L \gamma^m \bar{\partial} \theta_L) \tag{112}$$

- Using the standard definition for the momentum

$$p^m = \oint dz J^m - \oint d\bar{z} \bar{J}^m \tag{113}$$

one gets $p^i = \sqrt{2}\alpha_0^i$, and

$$p^- = \frac{1}{2p^+} \sum_n \left[a_{-n}^i a_n^i + \bar{a}_{-n}^i \bar{a}_n^i + n S_{-n}^a S_n^a + n \bar{S}_{-n}^{\hat{a}} \bar{S}_n^{\hat{a}} \right] \tag{114}$$

- Therefore, using that $a_0^i = \bar{a}_0^i$, the mass operator for the closed superstring states takes the form

$$M^2 = 2 \sum_{n=1}^{\infty} \left[a_{-n}^i a_n^i + \bar{a}_{-n}^i \bar{a}_n^i + n S_{-n}^a S_n^a + n \bar{S}_{-n}^{\hat{a}} \bar{S}_n^{\hat{a}} \right] \tag{115}$$

- Since the open superstring is only defined in the upper half-plane, the mass operator for open superstring states is given by

$$M^2 = \sum_{n=1}^{\infty}\left[a^i_{-n}a^i_n + nS^a_{-n}S^a_n\right] \tag{116}$$

- The massless spectrum is then found from the algebra satisfied by the fermionic zero modes. For the open superstring, this algebra is the same as the one found in the superparticle model, therefore

$$S^a_0|\dot{a}>= \frac{1}{\sqrt{2}}(\sigma^i)_{a\dot{a}}|i> \ , \ S^a_0|i>= \frac{1}{\sqrt{2}}(\sigma^i)_{a\dot{a}}|\dot{a}> \tag{117}$$

  so the ground state of the GS open superstring describes 10D super-Maxwell.
- This vacuum will be denoted by $|\dot{a}j>$ or, equivalently, $\mathbf{8}_v \oplus \mathbf{8}_c$.

Spectrum of open superstrings			
Mass	States	Bosons	Fermions
$M^2 = 0$	$\lvert \dot{a}, j >$	8	8
$M^2 = 1$	$a^i_{-1} \lvert \dot{a} j >,\ S^a_{-1} \lvert \dot{a} j >$	128	128
$M^2 = 2$	$a^i_{-2} \lvert \dot{a} j >, a^i_{-1} a^k_{-1} \lvert \dot{a} j >,$ $S^a_{-2} \lvert \dot{a} j >,\ S^a_{-1} S^b_{-1} \lvert \dot{a} j >,$ $S^a_{-1} a^i_{-1} \lvert \dot{a} j >$	1152	1152

Table 1: Spectrum of open superstrings.

- The massless spectrum of closed superstrings depends on the type of theory we are studying. They can easily be found from a tensor product of two massless open superstring states:

$$\text{Type IIB} \quad : \quad (\mathbf{8}_v \oplus \mathbf{8}_c) \otimes (\mathbf{8}_v \oplus \mathbf{8}_c) \tag{118}$$

$$\text{Type IIA} \quad : \quad (\mathbf{8}_v \oplus \mathbf{8}_c) \otimes (\mathbf{8}_v \oplus \mathbf{8}_s) \tag{119}$$

- Using standard group theory arguments, one can write (118)-(119) in terms of $SO(8)$ irrep.

$$\begin{aligned} \text{Type IIB} &: \quad (\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35} \oplus \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35})_B \oplus (\mathbf{8}_s \oplus \mathbf{8}_s \oplus \mathbf{56}_s \oplus \mathbf{56}_s)_F \\ \text{Type IIA} &: \quad (\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35} \oplus \mathbf{8} \oplus \mathbf{56})_B \oplus (\mathbf{8}_s \oplus \mathbf{8}_c \oplus \mathbf{56}_s \oplus \mathbf{56}_c)_F \end{aligned} \tag{120}$$

- Spectrum of Type IIB superstrings

Mass	States	Bosons	Fermions
$M^2 = 0$	$\phi_0 \otimes \tilde{\phi}_0$	128	128
$M^2 = 4$	$a^i_{-1}\phi_0 \otimes \bar{a}^j_{-1}\tilde{\phi}_0$, $S^a_{-1}\phi_0 \otimes \bar{S}^b_{-1}\tilde{\phi}_0$, $a^i_{-1}\phi_0 \otimes \bar{S}^a_{-1}\tilde{\phi}_0$, $S^a_{-1}\phi_0 \otimes \bar{a}^j_{-1}\tilde{\phi}_0$	32728	32728
$M^2 = 8$	$a^i_{-2}\phi_0 \otimes \bar{a}^j_{-1}\tilde{\phi}_0, \ldots$	2654208	2654208

Table 2: Spectrum of Type IIB superstrings.

- The light-cone gauge is preserved by the supersymmetry transformations:

$$\delta S^a_L = \sqrt{2p^+}\eta^a_L, \qquad \delta S^a_L = i\sqrt{\frac{1}{p^+}}\partial X^i(\sigma^i)^{a\dot{a}}\varepsilon_{L\,\dot{a}}$$

$$\delta S^{\hat{a}}_R = \sqrt{2p^+}\eta^{\hat{a}}_R, \qquad \delta S^{\hat{a}}_R = i\sqrt{\frac{1}{p^+}}\bar{\partial}X^i(\sigma^i)^{a\dot{a}}\varepsilon_{R,\dot{a}}$$

$$\delta X^i = 0, \qquad \delta X^i = -i\sqrt{\frac{1}{p^+}}(S_L\sigma^i\varepsilon_L) - i\sqrt{\frac{1}{p^+}}(S_R\sigma^i\varepsilon_R) \tag{121}$$

which are nothing but a combination of the original supersymmetry and kappa transformations.

- In addition, the light-cone gauge can be broken by a Lorentz transformation since

$$\delta\theta^\alpha = \frac{1}{2}[\frac{1}{2}\epsilon_{ij}(\gamma^{ij})^\alpha{}_\beta + \epsilon_{-j}(\gamma^{-j})^\alpha{}_\beta + \epsilon_{+j}(\gamma^{+j})^\alpha{}_\beta + \epsilon_{+-}(\gamma^{+-})^\alpha{}_\beta]\theta^\beta$$

and $(\gamma^+\gamma^{-j}\theta) \neq 0$.

- However, it is always possible to perform a kappa transformation such that the light-cone gauge is respected.
- This implies one should redefine the Lorentz generator J^{-i} so that the modified Lorentz transformation acts on S_L^a as

$$\delta S_L^a = -\frac{1}{2}\epsilon_{+-}S_L^a + \frac{1}{2}\epsilon_{i-}\frac{\partial X^i}{\partial X^+}S_L^a - \frac{1}{2}\epsilon_{i-}\frac{\partial X^j}{\partial X^+}(\sigma^j)^{a\dot{a}}(\sigma^i)_{\dot{a}b}S_L^b \tag{122}$$

- One can now compute the light-cone gauge Lorentz generators, using the standard Noether procedure. The result is

$$\begin{aligned} M^{mn}(z) = {} & 2i(X^{[m}\partial X^{n]}) + 2X^{[m}(\theta_R\gamma^{n]}\partial\theta_R) - \frac{1}{2}\partial X^p(\theta_L\gamma_p\gamma^{mn}\theta_L) \\ & + \frac{i}{4}(\theta_L\gamma^p\partial\theta_L)(\theta_L\gamma_p\gamma^{mn}\theta_L) + \frac{i}{4}(\theta_R\gamma^p\partial\theta_R)(\theta_R\gamma_p\gamma^{mn}\theta_R) \\ & + \frac{i}{2}(\theta_R\gamma^p\partial\theta_R)(\theta_L\gamma_p\gamma^{mn}\theta_L) \end{aligned} \tag{123}$$

and similarly for $\bar{M}^{mn}(\bar{z})$.

- The Lorentz charges are then computed to be

$$J^{mn} = x^{[m}p^{n]} - \frac{i}{2}\sum_{k=1}^{\infty}(\alpha_{-k}^{m}\alpha_{k}^{n} - \alpha_{-k}^{n}\alpha_{k}^{m}) + N_F^{mn} \qquad (124)$$

- The only non-zero contribution of N_F^{mn} comes from

$$N_F^{ij} = -\frac{i}{4}\sum_{k}(S_{-k}\sigma^{ij}S_k) \qquad (125)$$

  Therefore, the N_F^{i-} piece of the modified Lorentz transformation preserving the light-cone gauge (122) is a pure kappa transformation.

- One can check that the charge generating the transformation (122) takes the form

$$N^{i-} = \frac{1}{8}\oint \frac{\partial X^i}{\partial X^+}(S\sigma^{ij}S) \qquad (126)$$

- All these ingredients allow us to show that $[J^{i-}, J^{j-}] = 0$ if and only if $D = 10$.

Summary

- The 10D BS superparticle exhibits manifest supersymmetry and hides a local fermionic symmetry called (Siegel) kappa symmetry.
- The nature of its constraints (first- and second-class) does not allow us to separate them out in a simple way respecting Lorentz covariance.
- The superparticle spectrum is easily calculated in light-cone gauge and it describes the physical degrees of freedom of 10D super-Maxwell.
- The GS superstring can have at most $N = 2$ supersymmetries and it can be constructed in $D = 3, 4, 6, 10$ dimensions.
- The nature of its constraints does not allow us to quantize it in a Lorentz covariant manner. Light-cone gauge analysis is simple, and the physical spectrum is easily obtained and shown to be supersymmetric.
- To preserve the Lorentz algebra at quantum level, the spacetime dimension must be 10.

Lecture 3

10D Super-Yang-Mills in Superspace

- As we saw before, the (N=1) 10D superspace is described by the coordinates X^m, θ^α.
- Using these coordinates, one can define the operators

$$Q_\alpha = \partial_\alpha - i(\gamma^m\theta)_\alpha\partial_m \tag{127}$$

which realize the SUSY algebra $\{Q_\alpha, Q_\beta\} = -2i(\gamma^m)_{\alpha\beta}\partial_m$.
- The supersymmetric derivatives can then be introduced as

$$D_\alpha = \partial_\alpha + i(\gamma^m\theta)_\alpha\partial_m \tag{128}$$

and satisfy $\{D_\alpha, D_\beta\} = 2i(\gamma^m)_{\alpha\beta}\partial_m$,$\{D_\alpha, Q_\beta\} = 0$.
- These objects can be written in a more compact notation by introducing the so-called (super)vielbein fields ${E_A}^M$, ${E_M}^A$ satisfying ${E_A}^M {E_M}^B = \delta_A^B$, ${E_A}^N {E_M}^A = \delta_M^N$. Then,

$$D_A = {E_A}^M\partial_M \tag{129}$$

where $\partial_M = (\partial_m, \partial_\alpha)$.

- We then define the 1-form basis

$$E^A = dZ^M {E_M}^A \tag{130}$$

which will allow us to implement SUSY transformations as coordinate transformations in superspace.

- Explicitly, the matrix ${E_M}^A$ takes the form

$${E_M}^A \;=\; \begin{pmatrix} \delta_n^m & 0 \\ -i(\gamma^m\theta)_\alpha & \delta_\alpha^\beta \end{pmatrix} \tag{131}$$

- We can now introduce a 1-form (super)connection and define

$$\nabla = d + \mathbb{A} \tag{132}$$

where $\mathbb{A}$ is Lie-algebra valued.

- The (super)field-strength is then defined in the usual way

$$\mathbb{F} \;=\; d\mathbb{A} + \mathbb{A} \wedge \mathbb{A} \tag{133}$$

- It is not hard to see that $\mathbb{F}$ must satisfy the super-Bianchi identities

$$\nabla \mathbb{F} \;=\; 0 \tag{134}$$

- Explicitly, the 2-form superfield $\mathbb{F}$ can be written as follows

$$\mathbb{F} = \frac{1}{2} E^B E^A \mathbb{F}_{AB} = d(E^B \mathbb{A}_B) + E^B E^A \mathbb{A}_A \mathbb{A}_B \tag{135}$$

  which implies

$$\mathbb{F}_{AB} \;=\; 2D_{[A}\mathbb{A}_{B\}} + 2\mathbb{A}_A \mathbb{A}_B + T_{AB}{}^C \mathbb{A}_C \tag{136}$$

  where $T^A = dE^A$ is the so-called (super)torsion.

- Using eqn. (131) one can show that the only non-zero component of T^A is given by $T_{\alpha\beta}{}^m = -2i(\gamma^m)_{\alpha\beta}$.

- Therefore, one has

$$\mathbb{F}_{\alpha\beta} = D_\alpha \mathbb{A}_\beta + D_\beta \mathbb{A}_\alpha + \{\mathbb{A}_\alpha, \mathbb{A}_\beta\} - 2i(\gamma^m)_{\alpha\beta}\mathbb{A}_m \quad (137)$$

$$\mathbb{F}_{m\alpha} = \partial_m \mathbb{A}_\alpha - D_\alpha \mathbb{A}_m + [\mathbb{A}_m, \mathbb{A}_\alpha] \quad (138)$$

$$\mathbb{F}_{mn} = \partial_m \mathbb{A}_n - \partial_n \mathbb{A}_m + [\mathbb{A}_m, \mathbb{A}_n] \quad (139)$$

- Or, equivalently

$$\mathbb{F}_{\alpha\beta} = \{\nabla_\alpha, \nabla_\beta\} - 2i(\gamma^m)_{\alpha\beta}\nabla_m \quad (140)$$

$$\mathbb{F}_{m\alpha} = [\nabla_m, \nabla_\alpha] \quad (141)$$

$$\mathbb{F}_{mn} = [\nabla_m, \nabla_n] \quad (142)$$

- Using similar manipulations, the Bianchi identities in component form read

$$\nabla_{[A}\mathbb{F}_{BC\}} + T_{[AB}{}^{D}\mathbb{F}_{|D|C\}} = 0 \quad (143)$$

- Explicitly,

$$
\begin{aligned}
\nabla_{(\alpha}\mathbb{F}_{\beta\delta)} + 2i(\gamma^m)_{(\alpha\beta}\mathbb{F}_{\delta)m} &= 0 \qquad (144)\\
\nabla_m\mathbb{F}_{\alpha\beta} + 2\nabla_{(\alpha}\mathbb{F}_{\beta)m} - 2i(\gamma^n)_{\alpha\beta}\mathbb{F}_{nm} &= 0 \qquad (145)\\
2\nabla_{[m}\mathbb{F}_{n]\alpha} + \nabla_\alpha\mathbb{F}_{mn} &= 0 \qquad (146)\\
\nabla_{[m}\mathbb{F}_{np]} &= 0 \qquad (147)
\end{aligned}
$$

- To solve these identities, one needs to impose constraints.
- **Conventional constraint**:

$$
(\gamma^m)^{\alpha\beta}\mathbb{F}_{\alpha\beta} = 0 \qquad (148)
$$

This constraint kills one spin-$\frac{1}{2}$ field present in the spectrum. Indeed, one has

$$
\begin{aligned}
\mathbb{F}_{\alpha\beta} &= (\gamma^m)_{\alpha\beta}\mathbb{F}_m + (\gamma^{mnpqr})_{\alpha\beta}\mathbb{F}_{mnpqr}\\
\mathbb{F}_{m\alpha} &= \tilde{\mathbb{F}}_{m\alpha} + (\gamma_m)_{\alpha\beta}\mathbb{W}^\beta \qquad (149)
\end{aligned}
$$

where $\tilde{\mathbb{F}}_{m\alpha}$ is γ-traceless, and

$$
\mathbb{F}_m = f_m^{(0)} + \theta^\alpha f_{m\alpha}^{(1)} + \ldots \qquad (150)
$$

- Using the same decomposition for $f^{(1)}_{m\alpha}$, one finds it contains a spin-$\frac{1}{2}$ field λ^α of the same mass dimension as χ^α ($\frac{3}{2}$), the zeroth θ-component of $\mathbb{W}^\alpha$. Therefore, eqn. (148) reduces the number of spin-$\frac{1}{2}$ fermions to one.
- Notice that this constraint can always be satisfied by performing a field redefinition, namely $\mathbb{A}_A = (\mathbb{A}_\alpha, \mathbb{A}_m - \frac{i}{32}(\gamma_m)^{\alpha\beta}\mathbb{F}_{\alpha\beta})$.
- **Dynamical constraint**:

$$(\gamma^{mnpqr})^{\alpha\beta}\mathbb{F}_{\alpha\beta} = 0 \tag{151}$$

  This is the constraint which puts the theory on-shell.
- All in all, 10D SYM is described by setting

$$\mathbb{F}_{\alpha\beta} = 0 \tag{152}$$

- The Bianchi identities can then be cast as

-

$$
\begin{aligned}
(\gamma^m)_{(\alpha\beta}\mathbb{F}_{\delta)m} &= 0 && (153)\\
\nabla_{(\alpha}\mathbb{F}_{\beta)m} + i(\gamma^n)_{\alpha\beta}\mathbb{F}_{mn} &= 0 && (154)\\
2\nabla_{[m}\mathbb{F}_{n]\alpha} + \nabla_\alpha\mathbb{F}_{mn} &= 0 && (155)\\
\nabla_{[m}\mathbb{F}_{np]} &= 0 && (156)
\end{aligned}
$$

- Eqn. (153) sets $\tilde{\mathbb{F}}_{m\alpha} = 0$, and so

$$
\mathbb{F}_{m\alpha} = (\gamma_m)_{\alpha\beta}\mathbb{W}^\beta \qquad (157)
$$

- Eqn. (154) will provide a relation between $\mathbb{W}^\alpha$, $\mathbb{F}_{mn}$. Indeed, the 1-form and 5-form components of (154) must vanish, and so

$$
\begin{aligned}
&\text{1-form} \quad \rightarrow \quad \mathbb{C} = 0 \ , \ \mathbb{F}_{mn} = 2i\mathbb{C}_{mn}\\
&\text{5-form} \quad \rightarrow \quad \mathbb{C}_{mnpq} = 0 && (158)
\end{aligned}
$$

where

$$
\nabla_\alpha\mathbb{W}^\beta = \delta_\alpha^\beta\mathbb{C} + (\gamma^{mn})_\alpha{}^\beta\mathbb{C}_{mn} + (\gamma^{mnpq})_\alpha{}^\beta\mathbb{C}_{mnpq} \qquad (159)
$$

- Therefore,

$$\nabla_\alpha \mathbb{W}^\beta = -\frac{i}{2}(\gamma^{mn})_\alpha{}^\beta \mathbb{F}_{mn} \tag{160}$$

- This equation implies that the (n+1)-th θ term in $\mathbb{W}^\alpha$ is related to the n-th θ term in $\mathbb{F}_{mn}$.
- In addition, eqn. (155) can be used to demonstrate that no new fields will appear in the θ-expansion of $\mathbb{W}^\alpha$.
- Finally, eqn. (156) is just the usual Bianchi identity. Therefore we have the right field content of 10D SYM.
- The equations of motion immediately follow from (155), (160) and the dynamical constraint $\{\nabla_\alpha, \nabla_\beta\} = 2i(\gamma^m)_{\alpha\beta}\nabla_m$. Explicitly,

$$(\gamma^m)_{\alpha\beta}\nabla_m \mathbb{W}^\beta = 0 \tag{161}$$

$$\nabla_m \mathbb{F}^{mn} = \frac{i}{2}\gamma^n_{\alpha\beta}\{\mathbb{W}^\alpha, \mathbb{W}^\beta\} \tag{162}$$

- In a very similar way, one can describe 10D SYM by using the gauge field $\mathbb{A}_A$ and its gauge transformation. After removing spurious terms, the gauge potential possesses the following θ-expansion

$$\begin{aligned} \mathbb{A}_\alpha &= i(\gamma^m\theta)_\alpha a_m - \frac{1}{36}(\theta\gamma^{mnp}\theta)(\gamma_{mnp}\chi)_\alpha + \dots && (163) \\ \mathbb{A}_m &= a_m + i(\theta\gamma^m\chi) + \dots && (164) \end{aligned}$$

where $\delta a_m = \partial_m\lambda + [\lambda, a_m]$.

- One can check that coordinate transformations on superfields indeed induce SUSY transformations. For instance,

$$\delta_Q \mathbb{W}^\alpha = \epsilon^\beta Q_\beta(\mathbb{W}^\alpha) \quad (165)$$

After expanding in components, one finds

$$\delta\chi^\alpha = -\frac{i}{2}(\epsilon\gamma^{mn})^\alpha F_{mn} \quad (166)$$

as desired.

- In order to make contact with what we discussed in the BS superparticle context, let us write down the abelian version of the dynamical constraint we just studied:

$$D_\alpha \mathbb{A}_\beta + D_\beta \mathbb{A}_\alpha = 2i(\gamma^m)_{\alpha\beta}\mathbb{A}_m \quad (167)$$

- In light-cone gauge these equations reduce to the equations satisfied by physical spectrum of the BS superparticle. For simplicity, let us assume the only non-zero component of the momentum is k^+. One then has $\mathbb{A}^- = 0$. Therefore,

$$\begin{aligned} D_a \mathbb{A}_b + D_b \mathbb{A}_a &= -2\sqrt{2}i\delta_{ab}\mathbb{A}^+ & (168)\\ D_a \mathbb{A}_{\dot{b}} + D_{\dot{b}} \mathbb{A}_a &= 2i(\sigma^i)_{a\dot{b}}\mathbb{A}_i & (169)\\ D_{\dot{a}} \mathbb{A}_{\dot{b}} + D_{\dot{b}} \mathbb{A}_{\dot{a}} &= 0 & (170)\end{aligned}$$

Using the algebra,

$$\{D_a, D_b\} = -2\sqrt{2}i\delta_{ab}k^+ \ , \ \{D_a, D_{\dot{b}}\} = 0 \ , \ \{D_{\dot{a}}, D_{\dot{b}}\} = 0 \quad (171)$$

one learns that the component $\mathbb{A}_a$ is pure gauge ($\delta\mathbb{A}_\alpha = D_\alpha\Lambda$).

- In this way, one is left with

$$D_a \mathbb{A}_{\dot{a}} = 2i(\sigma^i)_{\dot{a}a} \mathbb{A}_i \qquad (172)$$

and $\mathbb{A}_{\dot{b}}(\theta^a)$.

- Similarly, one can use the e.o.m $\mathbb{F}_{m\alpha} = (\gamma_m \mathbb{W})_\alpha$ to show that

$$D_a \mathbb{A}_i = -\frac{\sqrt{2}}{2} k^+ (\sigma^i)_{a\dot{a}} \mathbb{A}_{\dot{a}} \qquad (173)$$

- Up to numerical coefficients, these are the equations of motion we found in the BS superparticle spectrum.

10D Pure Spinor Superparticles

- Let us take the gauge-fixed BS superparticle action

$$S = \int d\tau \left[P_m \dot{X}^m + \frac{i}{2} S^a \dot{S}^a - \frac{e}{2} P^2 \right] \tag{174}$$

  and add a new couple of conjugate fermionic variables (θ^α, p_β).

- In order to recover the original theory, one needs to add a fermionic symmetry that allows us to remove the new degrees of freedom introduced. This fermionic symmetry will be generated by

$$\hat{d}_\alpha = d_\alpha + \frac{1}{2^{\frac{1}{4}} \sqrt{P^+}} (\gamma^m \gamma^+ S)_\alpha P_m \tag{175}$$

  where $d_\alpha = p_\alpha + i(\gamma^m \theta)_\alpha P_m$. This constraint is first-class since

$$\{\hat{d}_\alpha, \hat{d}_\beta\} = -\frac{1}{P^+} (\gamma^+)_{\alpha\beta} P^2 \tag{176}$$

- Therefore the action can be rewritten as

$$S = \int d\tau \left[P_m \dot{X}^m + \frac{i}{2} S^a \dot{S}^a - \frac{e}{2} P^2 + p_\alpha \dot{\theta}^\alpha + f^\alpha \hat{d}_\alpha \right] \quad (177)$$

- After gauge-fixing $e = 1$, $f^\alpha = 0$, the BRST operator is given by

$$\hat{Q} = cP^2 + \hat{\lambda}^\alpha \hat{d}_\alpha - \frac{1}{2P^+} (\hat{\lambda}\gamma^+ \hat{\lambda}) b \quad (178)$$

- We will now show the BRST-cohomology of $\hat{Q}$ is equivalent to the BRST-cohomology of $Q = \lambda^\alpha d_\alpha$ with λ^α being a pure spinor satisfying $\lambda\gamma^m\lambda = 0$.
- To prove this, we first show that the $\hat{Q}$-cohomology is equivalent to the Q'-cohomology, where $Q' = \lambda'^\alpha \hat{d}_\alpha$ and $\lambda'\gamma^+\lambda' = 0$. Then, it will be demonstrated that the Q'-cohomology is equivalent to the Q-cohomology.

- The first equivalence follows from the following argument: Define $Q_0 = \lambda_0^\alpha \hat{d}_\alpha$. Then $Q_0^2 = -\frac{1}{2P^+}(\lambda_0\gamma^+\lambda_0)P^2$.
- Let V be a state such that $Q_0 V = (\lambda_0\gamma^+\lambda_0)W$. Therefore, $Q_0 W = -\frac{1}{2P^+}P^2 V$.
- Then, the state $\hat{V} = V - 2cP^+W$ is annihilated by $\hat{Q}$. In this manner, if a state V is BRST-closed under Q', there is always a state $\hat{V}$ which is BRST-closed under $\hat{Q}$.
- Now let us assume that there is a state V satisfying $V = Q_0\Omega + (\lambda_0\gamma^+\lambda_0)Y$, for some Y.
- One can then show that $\hat{Q}(\Omega + 2P^+cY) = \hat{V}$, where $\hat{V} = V - 2cP^+W$. Thus, if there is a state V which is BRST-exact under Q', there is always a state $\hat{V}$ which is BRST-exact under $\hat{Q}$.
- One can easily reverse the arguments and the first equivalence is proved.

- The constraint $(\lambda'\gamma^+\lambda') = 0$ implies that $(\gamma^+\lambda')^{\dot{a}}$ is an $SO(8)$ null spinor. Therefore it is left invariant under an $SU(4)$ subgroup. Under this $SU(4)$ subgroup, the chiral spinors $(\gamma^-\lambda')_a$, $(\gamma^+ d)_a$, S^a split into the $\mathbf{4}$, $\bar{\mathbf{4}}$ representations as follows

$$\begin{aligned}(\gamma^-\lambda')_a &= \left((\gamma^-\lambda')_A, (\gamma^-\lambda')_{\bar{A}}\right)\\ (\gamma^+ d)_a &= \left((\gamma^+ d)_A, (\gamma^- d)_{\bar{A}}\right)\\ S^a &= \left(S_A, S_{\bar{A}}\right)\end{aligned} \tag{179}$$

  where $A, \bar{A} = 1, \ldots, 4$.

- After performing the shift

$$S_A \rightarrow S_A - \frac{i}{4\cdot 2^{\frac{1}{4}}\sqrt{P^+}}(\gamma^+ d)_A \tag{180}$$

  the BRST operator will change by a similarity transformation.

- Explicitly,

$$Q' \quad \to \quad e^{-iS} Q' e^{iS} = Q' + i[Q,S] + \frac{1}{2}[[Q,S],S] + \dots \quad (181)$$

where $S = K S_{\bar{A}}(\gamma^+ d)_A$, and $K = -\frac{i}{4\cdot 2^{\frac{1}{4}}\sqrt{P^+}}$.

- Then, the first commutator is computed to be (assume only $P^+ \neq 0$):

$$[Q', S_{\bar{A}} d_A] \quad = \quad -\lambda'^{\alpha} S_{\bar{A}} \{d_\alpha, d_A\} + \frac{\sqrt{P^+}}{2^{\frac{1}{4}}} \lambda'^{a} \{S_a, S_{\bar{A}}\} d_A \quad (182)$$

- To evaluate this expression we split d_α, S^a into their $SU(4)$ components as follows

$$S_A = \frac{1}{\sqrt{2}}(S_{2a} + iS_{2a+1}) \qquad S_{\bar{A}} = \frac{1}{\sqrt{2}}(S_{2a} - iS_{2a+1})$$
$$d_A = \frac{1}{\sqrt{2}}(d_{2a} + id_{2a+1}) \qquad d_{\bar{A}} = \frac{1}{\sqrt{2}}(d_{2a} - id_{2a+1})$$
$$(183)$$

- It is not hard to see these variables satisfy the algebra

$$\{S_A, S_{\bar{A}}\} = 2\eta_{A\bar{A}} \quad , \quad \{d_A, d_{\bar{A}}\} = -4\sqrt{2}\eta_{A\bar{A}} P^+ \quad (184)$$

- Hence,

$$-iK\sqrt{2}[Q', S_A d_{\bar{A}}] \;=\; -\sqrt{2\sqrt{2}P^+}\lambda'_A S_{\bar{A}} - \lambda'_{\bar{A}} d_A \qquad (185)$$

- Notice that this implies $[[Q', S], S] = 0$, and so all the other contributions in the BCH formula vanish.
- Then, we are left with

$$\begin{aligned} Q' \;&\rightarrow\; Q' - iK\sqrt{2}[Q', S_A d_{\bar{A}}] \\ &\rightarrow\; \lambda'^{\dot{a}} d_{\dot{a}} + \lambda'_A d_{\bar{A}} + \sqrt{2\sqrt{2}P^+}\lambda'_A S_{\bar{A}} \end{aligned} \qquad (186)$$

  where $\lambda'^{\dot{a}}$ is null. If we define $\lambda^\alpha = (\lambda'^{\dot{a}}, \lambda'_A, 0)$, one can write

$$Q' \;\rightarrow\; \lambda^\alpha d_\alpha + \sqrt{2\sqrt{2}P^+}\lambda'_A S_{\bar{A}} \qquad (187)$$

- Next, we use the so-called quartet argument which states that the cohomology of a BRST operator q is the same as the cohomology of a BRST operator $q' = q + cb$, where (a, b) and (c, d) are bosonic and fermionic conjugate variables, respectively.

- Using this result, one learns that

$$Q' \quad \to \quad Q = \lambda^\alpha d_\alpha \tag{188}$$

where λ^α is a pure spinor satisfying $\lambda\gamma^m\lambda = 0$, and the action takes the form

$$S \quad = \quad \int d\tau \left[P_m \dot{X}^m + p_\alpha \dot{\theta}^\alpha + w_\alpha \dot{\lambda}^\alpha - \frac{1}{2} P^2 \right] \tag{189}$$

- In this manner, we have shown that the pure spinor superparticle defined by (189), (188) is physically equivalent to the semi-light-cone 10D BS superparticle.
- In this pure spinor framework, the problem of covariant quantization is translated into a cohomological problem.

- To find the physical spectrum, we write down the wavefunction in a coordinate representation as

$$\Psi(x,\theta,\lambda) = \Psi_0(x,\theta) + \Psi_1(x,\theta,\lambda) + \ldots \tag{190}$$

where the subscript indicates the polynomial degree in λ.
- Let us focus on the ghost number one sector, that is $\Psi_1(x,\theta,\lambda) = \lambda^\alpha V_\alpha(x,\theta)$.
- The physical state conditions then require

$$\begin{aligned} Q\Psi_1 = 0 \quad &\rightarrow \quad \lambda^\alpha \lambda^\beta D_\alpha V_\beta = 0 \\ &\rightarrow \quad (\gamma^{mnpqr})^{\alpha\beta} D_\alpha V_\beta = 0 \end{aligned} \tag{191}$$

and also,

$$\delta\Psi_1 = Q\Omega \quad \rightarrow \quad \delta V_\alpha = D_\alpha \Omega \tag{192}$$

for some arbitrary parameter Ω.
- Eqn. (191) is nothing but the dynamical constraint studied in the previous section, and eqn. (192) is the usual gauge transformation for the fermionic gauge potential.

- Therefore we identify

$$V_\alpha(x,\theta) = \mathbb{A}_\alpha(x,\theta) \tag{193}$$

- In this manner the 10D pure spinor superparticle describes 10D super-Maxwell in a manifestly supersymmetric way through the ghost number one state

$$\Psi_1(x,\theta,\lambda) = \lambda^\alpha \mathbb{A}_\alpha(x,\theta) \tag{194}$$

- One can use the gauge transformation (192) to fix the HS gauge: $\theta^\alpha \mathbb{A}_\alpha = 0$, in which $\mathbb{A}_\alpha$ look like

$$\begin{aligned}\mathbb{A}_\alpha(x,\theta) = {} & \frac{1}{2}(\theta\gamma^m)_\alpha A_m - \frac{1}{3}(\theta\gamma^m)_\alpha(\chi\gamma_m\theta) \\ & -\frac{1}{32}(\theta\gamma_p)_\alpha(\theta\gamma^{mnp}\theta)F_{mn} + \frac{1}{60}(\theta\gamma_m)_\alpha(\theta\gamma^{mnp}\theta)(\partial_n\chi\gamma_p\theta) \\ & +\frac{1}{1152}(\gamma_m\theta)_\alpha(\theta\gamma^{mrs}\theta)(\theta\gamma_s{}^{pq}\theta)\partial_r F_{pq} + \dots\end{aligned} \tag{195}$$

- It turns out that one also finds non-trivial cohomology at ghost numbers 0, 2 and 3.
- The ghost number 0 sector will describe a constant which can be identified as the ghost field of the BV description of 10D super-Maxwell.
- The ghost number 2 sector will describe the antifields of 10D super-Maxwell through the superfield $\mathbb{A}_{\alpha\beta} = (\gamma^{mnpqr})_{\alpha\beta}\mathbb{A}_{mnpqr}$ satisfying

$$\begin{aligned} \lambda^{\alpha}(\lambda\gamma^{mnpqr}\lambda)D_{\alpha}\mathbb{A}_{mnpqr} &= 0 \\ \delta\mathbb{A}_{mnpqr} &= D\gamma_{mnpqr}\Lambda \end{aligned} \tag{196}$$

where Λ_{α} is an arbitrary gauge parameter.
- The ghost number 3 sector describes a scalar through the structure

$$(\lambda\gamma_m\theta)(\lambda\gamma_n\theta)(\lambda\gamma_p\theta)(\theta\gamma^{mnp}\theta)c^* \tag{197}$$

which is identified with the ghost antifield.

- Therefore, the 10D pure spinor superparticle describes 10D super-Maxwell in a manifestly super-Poincaré covariant way, in its BV description.
- The $\lambda^3\theta^5$ structure will play an important role when computing amplitudes. It can be seen as the pure spinor analogue of the scalar $c\partial c\partial^2 c$ in RNS.
- Then, 10D super-Yang-Mills can be described by generalizing our previous construction as

$$S = Tr \int d^{10}x \langle \frac{1}{2}\Psi Q\Psi + \frac{1}{3}\Psi\Psi\Psi \rangle \tag{198}$$

where $\langle \lambda^3\theta^5 \rangle = 1$.

- Indeed, the equations of motion and gauge transformations following from (198) read

$$Q\Psi + g\Psi\Psi = 0 \quad , \quad \delta\Psi = Q\Lambda + [\Psi, \Lambda] \tag{199}$$

which coincide with the physical conditions studied previously.

10D Pure Spinor Superstrings

- Using similar arguments as the ones given in the superparticle context, the Type II pure spinor superstring is defined by the action

$$S = \int d^2z \left[\partial X_m \bar{\partial} X^m + p_\alpha \bar{\partial}\theta^\alpha + w_\alpha \bar{\partial}\lambda^\alpha + \bar{p}_{\hat{\alpha}} \partial \bar{\theta}^{\hat{\alpha}} + \bar{w}_{\hat{\alpha}} \partial \bar{\lambda}^{\hat{\alpha}}\right] \tag{200}$$

  and the BRST operator

$$Q = \int dz \lambda^\alpha d_\alpha + \int d\bar{z} \bar{\lambda}^{\hat{\alpha}} \bar{d}_{\hat{\alpha}} \tag{201}$$

- Notice that the central charge vanishes. Indeed

$$c = 10 - 2 \times 16 + 2 \times 11 = 0 \tag{202}$$

- Furthermore, the ghost Lorentz current satisfies the Kac-Moody algebra

$$N^{mn}(z) N^{pq}(w) = \frac{6\eta^{m[p}\eta^{q]n}}{(z-w)^2} + \frac{2\eta^{m[p}N^{q]n} - 2\eta^{n[p}N^{q]m}}{z-w} \tag{203}$$

- Remarkably, this is exactly the contribution required in $N^{mn}_{p,\theta} = \frac{1}{2}(p\gamma^{mn}\theta)$ to reproduce the same Kac-Moody algebra generated by the RNS current $\psi^m\psi^n$.
- The massless physical spectrum is then easily found at ghost number (1,1) as

$$V = \lambda^\alpha \bar{\lambda}^{\hat{\alpha}} \mathbb{A}_{\alpha\hat{\alpha}} \tag{204}$$

where $\mathbb{A}_{\alpha\hat{\alpha}}$ satisfies the equations of motion

$$(\gamma^{mnpqr})^{\alpha\beta} D_\alpha \mathbb{A}_{\beta\hat{\alpha}} = (\gamma^{mnpqr})^{\hat{\alpha}\hat{\beta}} D_{\hat{\alpha}} \mathbb{A}_{\alpha\hat{\beta}} = 0 \tag{205}$$

- And gauge transformations

$$\begin{aligned} \delta\mathbb{A}_{\alpha\hat{\alpha}} &= D_\alpha \Omega_{\hat{\alpha}} + D_{\hat{\alpha}} \Omega_\alpha, \\ (\gamma^{mnpqr})^{\alpha\beta} D_\alpha \Omega_\beta &= (\gamma^{mnpqr})^{\hat{\alpha}\hat{\beta}} D_{\hat{\alpha}} \Omega_{\hat{\beta}} = 0 \end{aligned} \tag{206}$$

- These are nothing but the Type II supergravity equations of motion and gauge transformations in superspace.

- A simple way of seeing the operator V correctly describes the Type II supergravity degrees of freedom is by expressing $\mathbb{A}_{\alpha\hat{\alpha}} = \mathbb{A}_{\alpha}\mathbb{A}_{\hat{\alpha}}$, where the spinor superfields are the same as the ones studied in the pure spinor quantization of the superparticle. Therefore, the superfield $\mathbb{A}_{\alpha\hat{\alpha}}$ will contain the tensor product of the states in each spinor superfield, that is the NS-NS, NS-R, R-NS, R-R states.
- A similar analysis can be done for the hetetoric pure spinor supertring to conclude that $N = 1$ Super-Yang-Mills and $N = 1$ Supergravity are the massless states in the pure spinor BRST-cohomology.

Summary

- 10D super-Yang Mills can be described in superspace after imposing conventional and dynamical (or physical) constraints.
- The light-cone version of these superspace equations of motion coincides with the equations found in the light-cone quantization of the 10D BS superparticle.
- The 10D BS superparticle in semi-light-cone gauge can be shown to have the same BRST-cohomology as the pure spinor superparticle, and so they both are physically equivalent to each other.
- The pure spinor superparticle describes 10D super-Maxwell in its BV version in a manifestly super-Poincaré covariant way.
- 10D super-Yang-Mills can be described in the pure spinor framework by a Chern-Simons-like action.
- The pure spinor superstring possesses a vanishing central charge and its quantization provides a manifestly supersymmetric description of the several string states. In particular, the massless levels of the pure spinor superstring describe Type II, Type I supergravity and 10D super-Yang-Mills in 10D superspace.

Lecture 4

Pure Spinor Superstring Scattering Amplitudes

- In the RNS formalism of open superstrings, the ghost number one state (at picture number zero) reads

$$V^{(-1)} = ce^{-\phi}\psi^m A_m e^{ik\cdot X} \tag{207}$$

- As seen before, this operator describes the gluonic state. It turns out that one can define an integrated vertex operator $\int dz\, U(z)$ such that

$$U = \{b, V\} \tag{208}$$

or, explicitly

$$U^{(-1)} = e^{-\phi}\psi^m A_m e^{ik\cdot X} \tag{209}$$

- Using that $\{Q, b\} = T$ and the standard Jacobi identity, it is easy to see that

$$\{Q, U\} = \partial V \tag{210}$$

- After bosonization (or fermionization), one can define the so-called picture charge. This quantum number is then used to write down vertex operators in different ways.
- At tree-level, one needs to saturate the 2 zero modes associated to the superconformal ghost γ, and so one can choose to have 2 picture number -1 vertex operators and N-2 picture number 0 vertex operators in the N-point correlator.
- The picture number 0 vertex operator for the gluonic state reads

$$V^{(0)} = c(\partial X^m A_m + \frac{1}{2}\psi^m\psi^n F_{mn})e^{ik\cdot X} \tag{211}$$

and its corresponding integrated version takes the form

$$U^{(0)} = (\partial X^m A_m + \frac{1}{2}\psi^m\psi^n F_{mn})e^{ik\cdot X} \tag{212}$$

- The N-point gluonic amplitude is then given by

$$\begin{aligned}\mathcal{A}(1,\ldots,N) &= \langle V_1^{(-1)}(0)V_2^{(-1)}(1)\int dz_3 U_3^{(0)}(z_3)\ldots \\ &\int dz_{N-1}U_{N-1}^{(0)}(z_{N-1})V_N^{(0)}(\infty)\rangle \end{aligned} \tag{213}$$

- In the pure spinor formalism, we have seen that the unintegrated vertex operator in the massless sector is described by

$$V = \lambda^{\alpha}\mathbb{A}_{\alpha} \tag{214}$$

- One can then use the relation $\{Q, U\} = \partial V$ to define an integrated vertex operator. Explicitly,

$$U = \partial\theta^{\alpha}\mathbb{A}_{\alpha} + \Pi^{m}\mathbb{A}_{m} + d_{\alpha}\mathbb{W}^{\alpha} + \frac{1}{2}N^{mn}\mathbb{F}_{mn} \tag{215}$$

- The prescription to compute scattering amplitudes is then proposed to be

$$\begin{aligned} \mathcal{A}(1,\ldots,N) = \langle V_1(0)V_2(1)\int dz_3 U_3(z_3)\ldots \\ \int dz_{N-1}U_{N-1}(z_{N-1})V_N(\infty)\rangle \end{aligned} \tag{216}$$

where $\langle\lambda^3\theta^5\rangle = 1$.

- It turns out that this prescription is supersymmetric and decouples spurious (BRST-exact) states from the physical amplitudes.

- Let us see some explicit examples. The 3-point function is given by

$$\mathcal{A}(1,2,3) \quad = \quad \langle V_1(z_1)V_2(z_2)V_3(z_3)\rangle + (2 \leftrightarrow 3) \qquad (217)$$

- Due to momentum conservation, there is not KN factor. To get the 3-gluon amplitude we can use the following distribution table

$A_{1\,\alpha}(\theta)$	$A_{2\,\alpha}(\theta)$	$A_{3\,\alpha}(\theta)$
1	1	3
1	3	1
3	1	1

- One then finds (up to normalization factors)

$$\mathcal{A}(1_B, 2_B, 3_B) = \Big[k_3^m A_{1\,r}A_{2\,s}A_{3\,n} - k_2^m A_{1\,r}A_{2\,n}A_{3\,s} + k_1^m A_{1\,n}A_{2\,r}A_{3\,s}\Big] \cdot\langle(\lambda\gamma^r\theta)(\lambda\gamma^s\theta)(\lambda\gamma_p\theta)(\theta\gamma^{pmn}\theta)\rangle$$

- After using the identity

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{qrs}\theta)\rangle = \frac{1}{120}\delta^{mnp}_{qrs} \tag{218}$$

one obtains

$$\mathcal{A}(1_B,2_B,3_B) = \Big[(A_1\cdot A_2)(A_3\cdot k_2)+(A_1\cdot A_3)(A_2\cdot k_1) + (A_2\cdot A_3)(A_1\cdot k_3)\Big] \tag{219}$$

- The final result must be dressed with the standard Chan-Paton factors.
- In a similar manner, one can get the 2-gluino 1-gluon amplitude from the following theta distribution

$A_{1\,\alpha}(\theta)$	$A_{2\,\alpha}(\theta)$	$A_{3\,\alpha}(\theta)$
1	2	2
2	1	2
2	2	1

- The result can be cast as

$$\mathcal{A}_{1_B,2_F,3_F} = A_{1\,n}(\chi_2\gamma^r\chi_3)\langle(\lambda\gamma^n\theta)(\lambda\gamma^m\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mrp}\theta)\rangle$$
$$= A_{1\,m}(\chi_2\gamma^m\chi_3) \quad (220)$$

- Analogously, one can compute the 4-point function for closed string strates as follows

$$\mathcal{A}(1,2,3,4) = \langle V_1(z_1,\bar{z}_1)V_2(z_2,\bar{z}_2)V_3(z_3,\bar{z}_3)\int_{\mathbb{C}} d^2z_4 U(z_4,\bar{z}_4)\rangle \quad (221)$$

where $V(z,\bar{z}) = V(z)\bar{V}(\bar{z})$, $U(z,\bar{z}) = U(z)\bar{U}(z)$.

- One can use $SL(2,\mathbb{C})$ invariance to fix $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$.
- The relevant OPEs to be carried out involve the exponentials with Π^m (and $\bar{\Pi}^m$) and the terms proportional to d_α and N^{mn} in U (and $\bar{U}$) since,

$$N^{mn}(z)\lambda^\alpha(w) \rightarrow \frac{\alpha'}{4}\frac{(\lambda\gamma^{mn})^\alpha}{z-w} \;,\quad D_\alpha(z)\mathbb{V}(w) \rightarrow -\frac{\alpha'}{2}\frac{D_\alpha\mathbb{V}}{z-w} \quad (222)$$

- Then, one finds

$$\mathcal{A}(1,2,3,4) = (\frac{\alpha'}{2})^2 \int d^2 z_4 \left(\frac{F_{12}}{z_4} + \frac{F_{21}}{1-z_4} \right) \left(\frac{\bar{F}_{12}}{\bar{z}_4} + \frac{\bar{F}_{21}}{1-\bar{z}_4} \right) \cdot |z_4|^{-\frac{1}{2}\alpha' t} |1-z_4|^{-\frac{1}{2}\alpha' u} \quad (223)$$

where

$$F_{12} = i k_1^m \langle (\lambda \mathbb{A}_1)(\lambda \mathbb{A}_2)(\lambda \mathbb{A}_3) \mathbb{A}_{4\,m} \rangle + \langle \mathbb{A}_{1\,m} (\lambda \mathbb{A}_2)(\lambda \mathbb{A}_3)(\lambda \gamma^m \mathbb{W}_4) \rangle \quad (224)$$

and F_{21} can be obtained from F_{12} after exchanging $1 \leftrightarrow 2$.

- Using standard identities of Γ-functions, this amplitude can be written in the compact way

$$\mathcal{A}(1,2,3,4) = K_0 \bar{K}_0 \frac{\Gamma(-\frac{\alpha' t}{4}) \Gamma(-\frac{\alpha' s}{4}) \Gamma(-\frac{\alpha' u}{4})}{\Gamma(1+\frac{\alpha' t}{4}) \Gamma(1+\frac{\alpha' s}{4}) \Gamma(1+\frac{\alpha' u}{4})} \quad (225)$$

where $K_0 = \frac{1}{2}(u F_{12} + t F_{21})$.

- Explicitly,

$$\begin{aligned} K_0 &= \langle (\partial_m \mathbb{A}_{1\,n})(\lambda\mathbb{A}_2)\partial^m(\lambda\mathbb{A}_3)(\lambda\gamma^n\mathbb{W}_4)\rangle \\ &\quad -\frac{1}{2}\langle \partial^m(\lambda\mathbb{A}_1)\partial^n(\lambda\mathbb{A}_2)(\lambda\mathbb{A}_3)\mathbb{F}_{4\,mn}\rangle + (1\leftrightarrow 2) \end{aligned} \tag{226}$$

- After using superspace equations of motion and ignore BRST-exact contributions, this object can also be written as

$$K_0 = -\langle (\lambda\mathbb{A}_1)(\lambda\gamma^m\mathbb{W}_2)(\lambda\gamma^n\mathbb{W}_3)\mathbb{F}_{4\,mn}\rangle \tag{227}$$

- This expression encodes the kinematic factors for any scattering amplitude of 4 SYM physical states (gluons or gluinos). It can be shown the bosonic sector of this correlator reproduces the standard t_8-tensor contracted with the four field-strengths.

Non-minimal Pure Spinor Formalism

- Using the standard quartet argument, one can introduce two pairs of conjugate variables $(\bar{\lambda}_\alpha, \bar{w}^\beta)$, (r_α, s^β), where $\bar{\lambda}_\alpha$ is a pure spinor and r_α satisfies $\bar{\lambda}\gamma^m r = 0$, through the modification

$$Q = \int dz \left[\lambda^\alpha d_\alpha + r_\alpha \bar{w}^\alpha\right] \tag{228}$$

  so that the BRST-cohomology does not change.
- The action for the non-minimal pure spinor open superstring then looks like

$$S = \int d^2z \left[\partial X^m \bar{\partial} X_m + p_\alpha \bar{\partial}\theta^\alpha + w_\alpha \bar{\partial}\lambda^\alpha + \bar{w}^\alpha \bar{\partial}\bar{\lambda}_\alpha + s^\alpha \bar{\partial} r_\alpha\right] \tag{229}$$

- The tree-level scattering amplitude prescription is given by

$$\mathcal{A} = \langle \mathcal{N} V_1(z_1) V_2(z_2) V_3(z_3) \int dz_3 U_3(z_3) \ldots \int dz_N U_N(z_N)\rangle \tag{230}$$

- Explicitly,

$$\mathcal{A} = \int [d\lambda][d\bar{\lambda}][dr]d^{16}\theta \mathcal{N} \lambda^{\alpha}\lambda^{\beta}\lambda^{\delta} f_{\alpha\beta\delta}(\theta) \tag{231}$$

where,

$$[d\lambda]\,\lambda^{\alpha}\lambda^{\beta}\lambda^{\gamma} = \epsilon_{\rho_1\ldots\rho_{11}\kappa_1\ldots\kappa_5}\mathcal{T}^{(\alpha\beta\delta)[\kappa_1\ldots\kappa_5]}d\lambda^{\rho_1}\ldots d\lambda^{\rho_{11}} \tag{232}$$

$$\left[d\bar{\lambda}\right]\,\bar{\lambda}_{\alpha}\bar{\lambda}_{\beta}\bar{\lambda}_{\gamma} = \epsilon^{\rho_1\ldots\rho_{11}\kappa_1\ldots\kappa_5}\mathcal{T}_{(\alpha\beta\delta)[\kappa_1\ldots\kappa_5]}d\bar{\lambda}_{\rho_1}\ldots d\bar{\lambda}_{\rho_{11}} \tag{233}$$

$$[dr] = \epsilon_{\alpha_1\ldots\alpha_{11}\kappa_1\ldots\kappa_5}\mathcal{T}^{(\alpha\beta\gamma)[\kappa_1\ldots\kappa_5]}\bar{\lambda}_{\alpha}\bar{\lambda}_{\beta}\bar{\lambda}_{\gamma}\partial_r^{\alpha_1}\ldots\partial_r^{\alpha_{11}} \tag{234}$$

and,

$$\mathcal{N} = e^{\{Q,-(\bar{\lambda}\theta)\}} = e^{-\lambda\bar{\lambda}-r\theta} \tag{235}$$

- This non-minimal prescription is easily shown to be equivalent to the minimal one studied in the previous section.

Multiloop Prescription

- Using the non-minimal formalism, one can construct a b-ghost satisfying $\{Q, b\} = T$. It takes the form

$$\begin{aligned} b &= s^{\alpha}\partial\bar{\lambda}_{\alpha} + \frac{1}{4(\lambda\bar{\lambda})}\Big[2\Pi^{m}(\bar{\lambda}\gamma_{m}d) - N_{mn}(\bar{\lambda}\gamma^{mn}\partial\theta) - J_{\lambda}(\bar{\lambda}\partial\theta) \\ &\quad -\bar{\lambda}\partial^{2}\theta\Big] + \frac{(\bar{\lambda}\gamma^{mnp}r)}{192(\lambda\bar{\lambda})^{2}}\Big[(d\gamma_{mnp}d) + 24N_{mn}\Pi_{p}\Big] \\ &\quad -\frac{(r\gamma^{mnp}r)}{16(\lambda\bar{\lambda})^{3}}(\bar{\lambda}\gamma_{m}d)N_{np} + \frac{(r\gamma^{mnp}r)}{128(\lambda\bar{\lambda})^{4}}(\bar{\lambda}\gamma_{p}{}^{qr}r)N_{mn}N_{qr} \end{aligned} \tag{236}$$

- Using this b-ghost a g-loop amplitude prescription is formulated as

$$\mathcal{A} = d^{3g-3}\tau\langle\mathcal{N}(y)\prod_{i=1}^{3g-3}\left(\int dw_{i}\mu_{i}(w_{j})b(w_{j})\right)\prod_{j=1}^{N}\int dz_{j}U(z_{j})\rangle \tag{237}$$

- A conformal weight 1 field $\Phi(z)$ can be written in a g genus surface as

$$\Phi(z) = \hat{\Phi}(z) + \sum_{I=1}^{g} \Phi^I \Omega_I \tag{238}$$

where $\hat{\Phi}(z)$ have no zero modes and obeys $\int_{a_I} \hat{\Phi} = 0$, and Ω_I are the g holomorphic 1-forms satisfying $\int_{a_I} \Omega_J = \delta_{IJ}$.

- One then needs to integrate out the g zero modes corresponding to w_α, $\bar{w}_\alpha$, s^α, d_α. For such a purpose, one uses the measures

$$\begin{aligned}
\left[dw^I\right] &= (\lambda\gamma^m)_{\kappa_1}(\lambda\gamma^n)_{\kappa_2}(\lambda\gamma^p)_{\kappa_3}(\gamma_{mnp})_{\kappa_4\kappa_5}\epsilon^{\kappa_1\ldots\kappa_5\rho_1\ldots\rho_{11}} \\
&\quad \cdot dw^I_1 \ldots dw^I_{11} \\
\left[d\bar{w}^I\right] &= (\bar{\lambda}\gamma^m)^{\kappa_1}(\bar{\lambda}\gamma^n)^{\kappa_2}(\bar{\lambda}\gamma^p)^{\kappa_3}(\gamma_{mnp})^{\kappa_4\kappa_5}\epsilon_{\kappa_1\ldots\kappa_5\rho_1\ldots\rho_{11}} \\
&\quad \cdot d\bar{w}^I_1 \ldots d\bar{w}^I_{11} \\
\left[ds^I\right] &= (\lambda\bar{\lambda})^{-3}(\lambda\gamma^m)_{\kappa_1}(\lambda\gamma^n)_{\kappa_2}(\lambda\gamma^p)_{\kappa_3}(\gamma_{mnp})_{\kappa_4\kappa_5}\epsilon^{\kappa_1\ldots\kappa_5\rho_1\ldots\rho_{11}} \\
&\quad \cdot \partial^{s^I}_{\rho_1} \ldots \partial^{s^I}_{\rho_{11}}
\end{aligned} \tag{239}$$

- The regularization factor will be chosen as $\mathcal{N} = e^{-\lambda\bar{\lambda} - r\theta - w^I \bar{w}^I + s^I d^I}$
- Therefore the N-point g-loop correlator takes the form

$$\mathcal{A} = \int [d\lambda][d\lambda][dr] \prod_{I=1}^{g} [dw^I][d\bar{w}^I][ds^I](d^{16}d^I)d^{16}\theta \mathcal{N} f(\theta) \tag{240}$$

- This regularization scheme can be safely used up to two loops, since when $\lambda\bar{\lambda} \to 0$, the measure goes like

$$[d\lambda][d\lambda][dr] \prod_{I=1}^{g} [dw^I][d\bar{w}^I][ds^I](d^{16}d^I)d^{16}\theta \mathcal{N} \quad \to \quad \lambda^{8+3g}\bar{\lambda}^{11} \tag{241}$$

  and so the integrand must diverge slower than $\lambda^{-8-3g}\bar{\lambda}^{-11}$. As one needs to insert 3g-3 b-ghosts, and each b-ghost diverges as $\lambda^{-4}\bar{\lambda}^{-3}$, g=2 is the maximum number allowed.
- For higher-loops a different regularization scheme has been proposed and, in principle, it can be used for computing multiloop amplitudes.

- At 1-loop, one has the following useful result

$$\int d^{16}d[dw][d\bar{w}][ds]e^{-\lambda\bar{\lambda}-r\theta-w\bar{w}+sd}d_{\alpha_1}d_{\alpha_2}d_{\alpha_3}d_{\alpha_4}d_{\alpha_5}f^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}(r,\theta)$$
$$= (\lambda^3)_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}f^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}(D,\theta)$$

where $(\lambda^3)_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}$ is totally antisymmetric in its indices. Explicitly,

$$(\lambda^3)_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} = (\lambda\gamma^m)_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5} \tag{242}$$

- We can now easily compute the 4-point 1-loop amplitude. Since one needs to saturate 5 d_α zero modes and each vertex operator contains at most 1 d (through the term $d_\alpha\mathbb{W}^\alpha$), the b-ghost must contribute with 2 d's.
- This term has the form

$$b_{d^2} = \frac{1}{192(\lambda\bar{\lambda})^2}(d\gamma^{mnp}d)(\bar{\lambda}\gamma_{mnp}r) \tag{243}$$

- Therefore, one needs to compute

$$\int d^{16}\theta \int [d\lambda][d\bar{\lambda}][dr]e^{-\lambda\bar{\lambda}-r\theta}(\lambda\bar{\lambda})^{-2}(\lambda^4)(\bar{\lambda}\gamma^{mnp}D)\mathbb{A}\mathbb{W}\mathbb{W}\mathbb{W} \quad (244)$$

- Using $U(5)$ arguments or the result (242), one can show that the covariant expression for the integrand looks like

$$(\bar{\lambda}\gamma_{mnp}D)\Big[(\lambda\mathbb{A})(\lambda\gamma^m\mathbb{W})(\lambda\gamma^n\mathbb{W})(\lambda\gamma^p\mathbb{W})\Big] \quad (245)$$

- Using again $U(5)$ arguments or 10D pure spinor identities plus superspace equations of motion one can show that (up to an overall coefficient)

$$K_1 = (244) = \langle(\lambda\mathbb{A}_1)(\lambda\gamma^m\mathbb{W}_2)(\lambda\gamma^n\mathbb{W}_3)\mathbb{F}_{4\,mn}\rangle \quad (246)$$

- Analogously, one can use the non-minimal pure spinor prescription to compute the 4-point 2-loop open string amplitude. The kinematic factor is given by

$$K_2 \quad \propto \quad \langle(\lambda\gamma^{mnpqr}\lambda)\mathbb{F}_{1\,mn}\mathbb{F}_{2\,pq}\mathbb{F}_{3\,rs}(\lambda\gamma^{s}\mathbb{W}_4)\rangle \tag{247}$$

- Using superspace equations of motion and pure spinor identities, one learns that (up to an overall constant)

$$K_2 \quad = \quad s_{12}\langle(\lambda\mathbb{A}_1)(\lambda\gamma^{m}\mathbb{W}_2)(\lambda\gamma^{n}\mathbb{W}_3)\mathbb{F}_{4\,mn}\rangle \tag{248}$$

- Therefore we have shown that the tree-level, 1-loop and 2-loop kinematic factors for the 4-point function are proportional to each other. This result is valid for bosons and fermions, and component amplitudes can easily be extracted using the familiar pure spinor measure $\langle\lambda^3\theta^5\rangle = 1$.

Curved Backgrounds

- Pure spinor superstrings can easily be coupled to curved backgrounds. The requirements of nilpotency and holomorphicity of the BRST charge impose conventional and dynamical constraints the background must satisfy. Type IIA/IIB, Type I, N=1 SYM.
- α'-corrections modify these constraints in such a way that the pure spinor BRST charge keeps nilpotent at quantum level. Consistency with the GS mechanism.
- The fact one can couple pure spinor superstrings to curved background allows us in principle to study string theory in $AdS_5 \times S_5$ at quantum level, and so the pure spinor formalism is the most promising framework to understand better/prove the AdS/CFT conjecture.

Pure Spinor Master Actions

- Using the non-minimal formalism one can formulate a BV-like framework for pure spinor superfields.
- In this manner, pure spinor master actions have been constructed for several theories including N=1 super-Yang-Mills, N=1 super Born-Infeld (abelian and non-abelian), N=1 Supergravity, etc.
- Although the interaction terms in these actions are complicated functions of non-minimal variables, the pure spinor actions are remarkably much simpler than the component ones.
- It has been shown there exists a systematic procedure to extract equations of motion in ordinary superspace from these master actions in non-minimal pure spinor superspace.

11D Pure Spinors

- 11D pure spinors have been used to construct a pure spinor supermembrane which reduces to the standard Type IIA pure spinor superstring and the 11D pure spinor superparticle in the appropriate limits.
- The quantization of the 11D pure spinor superparticle describes the BV version of 11D linearized supergravity.
- In this framework, the 11D supergravity physical fields are located at ghost number three.
- A BRST-closed ghost number one vertex operator has been recently constructed and a relation with the ghost number three vertex operator has been proposed.
- M-Theory conjecture: Scattering amplitudes of this 11D supermembrane will contain non-perturbative information on superstring theory.

Other topics

- Twistors.
- Pure spinor chiral strings.
- Pure spinor QFTs.
- CY-compactifications.
- Matrix theory.
- ...

Summary

- The pure spinor formalism for superstrings produces compact expressions in pure spinor superspace involving interactions of bosons and fermions.
- The non-minimal pure spinor formalism allows us to construct a b-ghost satisfying the standard relation $\{Q, b\} = T$.
- Using pure spinor identities and superspace equations of motion, many scary-looking expressions can easily be manipulated and calculated in pure spinor superspace.
- The non-minimal formalism allows us to easily see (compared to the other formalisms) how the kinematic factors for the 4-point function at 0-, 1-, 2-loops are related to each other.
- Pure spinors have many other interesting applications in addition to the one we have seen in this minicourse. These include the study of superstrings in AdS5 $\times$ S5, stringy corrections for superspace constraints, QFT for maximally supersymmetric gauge theories, M-Theory, etc.